Solution to Problems ♡–6

Problem A: *Find all nonnegative integers n such there are integers a and b with the property*

$$n^2 = a + b \quad \textit{and} \quad n^3 = a^2 + b^2.$$

Answer: Suppose that n has the required property as witnessed by integers a, b. Note that

$$(a+b)^2 \leq (a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$$

and hence $(n^2)^2 \leq 2n^3$. Hence necessarily $n \leq 2$. Now we easily verify that

- $n = 0$ has the required property as witnessed by $a = b = 0$,
- $n = 1$ has the required property as witnessed by $a = 1$ and $b = 0$,
- $n = 2$ has the required property as witnessed by $a = b = 2$.

Consequently, the only integers with the property described in our problem are $0, 1, 2$.

Correct solution was received from :

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Problem B: *Find all primes p such that $p^2 + 11$ has exactly six different divisors (including 1 and the number itself).*

Answer: First note that for any prime p the product $(p-1)p(p+1)$ is divisible by 3 and hence if $p \neq 3$ then $3|(p^2 - 1)$. Consequently $3|(p^2 + 11)$ for all primes $p \neq 3$.

Also, if $p \neq 2$ then both $p - 1$ and $p + 1$ are even and $4|(p^2 - 1)$. Consequently $4|(p^2 + 11)$ for all primes $p \neq 2$.

Therefore, if the prime p is larger then 3, then $12|(p^2+11)$. Since 12 itself has six divisors (1, 2, 3, 4, 6, 12) and $p^2 + 11 > 12$ (for $p > 3$) we conclude that $p^2 + 11$ must have more than 6 divisors. Now we easily verify that

- if $p = 2$ then $p^2 + 11 = 15$ has exactly four divisors (1, 3, 5, 15), and
- if $p = 3$ then $p^2 + 11 = 20$ has exactly six divisors (1, 2, 4, 5, 10, 20).

Consequently, the only prime number p such that $p^2 + 11$ has exactly six different divisors is $p = 3$.

Correct solution was received from :

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