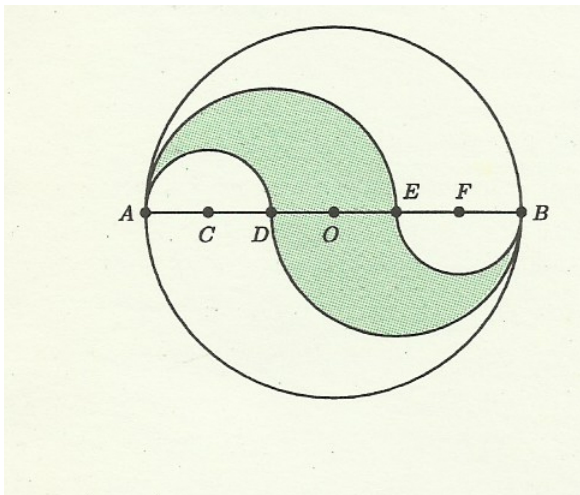
Solution to Problems ♡–10

Problem A: *Is it possible to divide a disk into three parts with equal areas and with perimeters the same as that of the disk ?*

Answer: Yes, you may do this for instance in the following way. Divide the diameter AB of the disk into six equal parts using points C, D, O, E and F. Semi-circles with diameters AD, AE, BE and BD will determine a partition of the disk into three regions, see the picture below:



Let r be the radius of the disk, $r = \frac{1}{2}|AB|$. The area of the shaded central region is

$$2 \cdot \left[\frac{1}{2}\pi\left(\frac{2}{3}r\right)^2 - \frac{1}{2}\pi\left(\frac{1}{3}r\right)^2\right] = \frac{1}{3}\pi r^2,$$

and its perimeter is

$$\pi \cdot \frac{2}{3}r + \pi \cdot \frac{1}{3}r + \pi \cdot \frac{2}{3}r + \pi \cdot \frac{1}{3}r = 2\pi r.$$

The two white regions are congruent and each has area

$$\frac{1}{2}\pi\left(\frac{1}{3}r\right)^2 + \frac{1}{2}\pi r^2 - \frac{1}{2}\pi\left(\frac{2}{3}r\right)^2 = \frac{1}{3}\pi r^2,$$

and their perimeters are

$$\pi \cdot \frac{1}{3}r + \pi \cdot r + \pi \cdot \frac{2}{3}r = 2\pi r.$$

Correct solution was received from :

(1) Cody Anderson — POW 10A: ♡

(2) Brad Tuttle — POW 10A: ♡

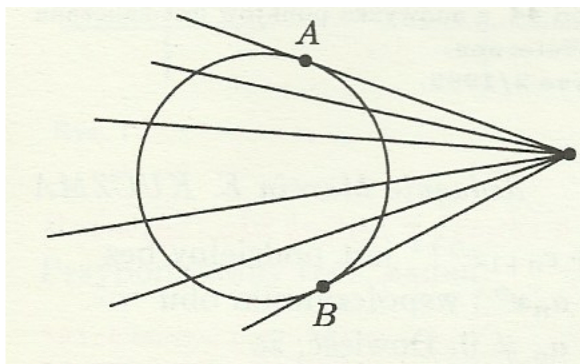
Problem B: *Is it possible to partition the three dimensional space* $\mathbb{R}^3$ *into pairwise disjoint circles ?*

Answer: Yes, it is possible. We will demonstrate this by first partitioning sphere without two points and an open ball with an extra point, and only then we will partition $\mathbb{R}^3$.

Step 1: *Sphere without two points*

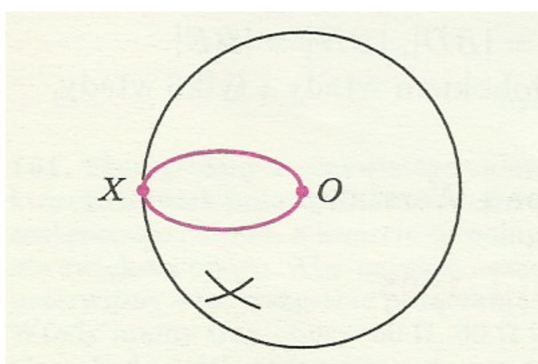
A sphere without two antipodal points (poles) can be partitioned into circles by parallels (circles of latitude).

A sphere without two non-antipodal points can be partitioned into disjoint circles by intersecting it with planes as in the picture below.



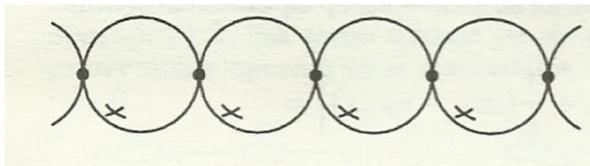
Step 2: *Open ball with adjoined point on the edge (boundary)*

Let O be the center of an open ball B and X be a point on the (spherical) boundary of the ball. We consider the set $B \cup \{X\}$ and our aim is to partition it into disjoint circles. First, we take a circle C with diameter XO and we remove it from the ball:



Looking at the set $B \cup \{X\} \setminus C = B \setminus C$ we see that it is the disjoint union of spheres centered at O, each included in the ball B and each having two points (intersection with C) removed. Each of these spheres without two points can be partitioned into disjoint circles by Step 1. Those partitions together with the circle C give the desired partition of $B \cup \{X\}$.

Step 3: *The three dimensional space* $\mathbb{R}^3$
Choose a sequence of open balls of radius 1 positioned on a straight line ℓ in such a manner that successive balls are tangent and together with the points of tangency the balls cover the line ℓ:



Add to each ball exactly one of the (two) tangency points. The open balls with these extra points can be partitioned into circles by Step 2. The remainder of the space $\mathbb{R}^3$ can be filled up with disjoint circles centered at points on ℓ and lying in the planes perpendicular to ℓ.

No correct solution were received