Problem A: Find the greatest lower and the least upper bounds of the set
\[ \left\{ \frac{(n + 1)^2}{2^n} : n \in \mathbb{N} \right\}. \]

Answer: First we are going to show the following three observations.

Claim A For every real number \( x \geq 4 \) we have
\[ (x + 1)^3 \leq 2x^3. \]

Proof of the Claim. Let \( f(x) = (x + 1)^3 \) and \( g(x) = 2x^3 \) for \( x \in \mathbb{R} \). Clearly \( f'(x) = 3(x + 1)^2 \) and \( g'(x) = 6x^2 \). Also

1. \( f(4) = 125 < 128 = g(4) \), and
2. \( f'(x) < g'(x) \) for \( x \geq 4 > 1 + \sqrt{2} \).

Therefore \( f(x) < g(x) \) for all \( x \geq 4 \) and our Claim easily follows. \( \square \)

Claim B For every natural number \( n \geq 11 \) we have
\[ (n + 1)^3 < 2^n. \]

Proof of the Claim. We show our Claim by induction on \( n \geq 11 \). Let \( P(n) \) be the assertion that the inequality holds for \( n \) and let us verify that the assumptions of the Theorem on Mathematical induction are satisfied by the formula \( P(n) \).

Basic Step \( n = 11 \)
By direct computation we check that \((11 + 1)^3 = 1728 < 2048 = 2^{11}\), so \( P(11) \) holds true indeed.

Inductive Step Let \( n \geq 11 \) and let us assume that \( P(n) \) holds true, that is we assume
\[ (*)^n (n + 1)^3 < 2^n. \]
We want to derive that then \( P(n + 1) \) is true. Using Claim A (for \( x = n + 1 \)) and then \((*)^n \) we get
\[ \left( (n + 1) + 1 \right)^3 \leq 2 \cdot (n + 1)^3 < 2 \cdot 2^n = 2^{n+1}. \]
Consequently, \( P(n + 1) \) holds true.
Thus the assumptions of the Theorem on Mathematical Induction are satisfied and we may conclude that \((\forall n \geq 11) \ P(n)\), as desired. □

**Claim C**  
*For every natural number \(n \geq 6\) we have*  
\[
(n + 1)^2 < 2^n.
\]

**Proof of the Claim.** We show our Claim by induction on \(n \geq 6\). Let \(P(n)\) be the assertion that the inequality holds for \(n\) and let us verify that the assumptions of the Theorem on Mathematical induction are satisfied by the formula \(P(n)\).

**Basic Step**  
\(n = 6\)  
By direct computation we check that \((6 + 1)^2 = 49 < 64 = 2^6\), so \(P(6)\) holds true indeed.

**Inductive Step** Let \(n \geq 6\) and let us assume that \(P(n)\) holds true, that is we assume  
\[
(\ast\ast)^n \ (n + 1)^2 < 2^n.
\]

We want to derive that then \(P(n + 1)\) is true. For this we note that for all \(n \geq 6\) we have \(2(n + 2) < 2^n\). Now, using \((\ast\ast)^n\), we get  
\[
\left((n+1)+1\right)^2 = (n+1)^2 + 2(n+1) + 1 < 2^n + 2(n + 2) < 2^n + 2^n = 2^{n+1}.
\]

Consequently, \(P(n + 1)\) holds true.

Thus the assumptions of the Theorem on Mathematical Induction are satisfied and we may conclude that \((\forall n \geq 6) \ P(n)\), as desired. □

It follows from Claim B that  
\[
0 < \frac{(n + 1)^2}{2^n} < \frac{(n + 1)^2}{(n + 1)^3} = \frac{1}{(n + 1)} \quad \text{for all} \ n \geq 11.
\]

Therefore 0 is the greatest lower bound of our set.

By Claim C we know that  
\[
\frac{(n + 1)^2}{2^n} < 1 \quad \text{for all} \ n \geq 6.
\]

The numbers \(2, \frac{9}{4}, \frac{25}{16}, \frac{36}{32}\) (greater than 1) also belong to our set. Thus the least upper bound of the set is \(\frac{9}{4}\).
Problem B: Show that for any irrational number $\alpha$ and for any positive integer $n$ there exist a positive integer $q_n$ and an integer $p_n$ such that

$$|\alpha - \frac{p_n}{q_n}| < \frac{1}{nq_n}.$$ 

Answer: Fix a natural number $n$ and consider the $n+1$ real numbers

$$0, \alpha - \lfloor \alpha \rfloor, 2\alpha - \lfloor 2\alpha \rfloor, \ldots, n\alpha - \lfloor n\alpha \rfloor.$$ 

Since $\alpha$ is irrational, these numbers must be distinct. Each of these numbers belongs to the interval $[0, 1)$. Since the $n$ intervals $\left[ \frac{j}{n}, \frac{j+1}{n} \right)$, $j = 0, 1, \ldots, n - 1$ cover $[0, 1)$, there must be one which contains at least two of these points, say $n_1\alpha - \lfloor n_1\alpha \rfloor$ and $n_2\alpha - \lfloor n_2\alpha \rfloor$ with $0 \leq n_1 < n_2 \leq n$. So

$$|n_2\alpha - \lfloor n_2\alpha \rfloor - n_1\alpha + \lfloor n_1\alpha \rfloor| < \frac{1}{n}$$

and dividing both sides of the inequality by $n_2 - n_1 > 0$ we get

$$\left| \alpha - \frac{\lfloor n_2\alpha \rfloor - \lfloor n_1\alpha \rfloor}{n_2 - n_1} \right| < \frac{1}{n(n_2 - n_1)}.$$ 

Thus it is enough to take $q_n = n_2 - n_1$ and $p_n = \lfloor n_2\alpha \rfloor - \lfloor n_1\alpha \rfloor$.

Correct solution were received from:

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