Solution to Problems ♠–5

Problem A:  Show that every number in the sequence

1007, 10017, 100117, 1001117, 10011117,...

is divisible by 53.

Answer:  For a natural number \( n \) let \( a_n \) be the natural number with the decimal expansion \( 1001 \underbrace{11\ldots 1}_{n-1}7 \). One should notice that

\[
a_{n+1} = (a_n - 6) \cdot 10 + 7 = 10a_n - 53.
\]

Now by a straightforward induction on \( n \in \mathbb{N} \) we show that each number \( a_n \) is divisible by 53. To this end we verify that the assumptions of the Theorem on Mathematical Induction are satisfied.

Basic Step:  \( n = 1 \)

We note that \( a_1 = 1007 = 19 \cdot 53 \), so our claim is readily true for \( n = 1 \).

Inductive Step:  Let \( n \in \mathbb{N} \) be an arbitrary natural number and let us assume that

(\( \oplus \)_0) \( a_n \) is divisible by 53.

Thus for some integer \( k \) we have

(\( \oplus \)_1) \( a_n = 53 \cdot k \).

Now,

\[
a_{n+1} = 10a_n - 53 = 10 \cdot 53 \cdot k - 53 = 53 \cdot (10k - 1).
\]

Since \( 10k - 1 \) is an integer, we conclude that (under our inductive assumption (\( \oplus \)_0), \( a_{n+1} \) is divisible by 53. Thus if our claim is true for \( n \), then it is also true for \( n + 1 \).

Consequently, by the Theorem on Mathematical Induction we may conclude that \((\forall n \in \mathbb{N})(53 \mid a_n)\).

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Problem B: Show that for $n \geq 6$ a square can be dissected into $n$ smaller squares, not necessarily all of the same size.

Answer: For a natural number $n \geq 8$ let $P(n)$ be the assertion that a square can be dissected into $k$ smaller squares (not necessarily all of the same size) for $k = n$ and for $k = n - 1$ and for $k = n - 2$. It should be clear that the sentence
\[ (\forall n \geq 8) \ P(n) \]
is equivalent to the claim that for $n \geq 6$ a square can be dissected into $n$ smaller squares, not necessarily all of the same size.

We will show $(\forall n \geq 8) \ P(n)$ using the Theorem on Mathematical Induction. To this end we will verify that the formula $P(n)$ satisfies the assumptions of this theorem.

Basic Step: $n = 8$
We have to justify that a square can be partitioned into 8, 7, and 6 squares (not necessarily all of the same size). But this follows by the following pictures.

Inductive Step: Let $n \geq 8$ be an arbitrary natural number and let us assume that $P(n)$ holds true, that is
\[ (\exists) \ a \ square \ can \ be \ dissected \ into \ k \ smaller \ squares \ (not \ necessarily \ all \ of \ the \ same \ size) \ for \ k = n \ and \ for \ k = n - 1 \ and \ for \ k = n - 2. \]
We are going to argue that then $P(n + 1)$ holds true. First we note that by our assumption $(\exists)$, a square can be divided into $n = (n + 1) - 1$ and into $n - 1 = (n + 1) - 2$ squares. To create a partition into $n + 1$ squares we use the the inductive hypothesis $(\exists)$ to divide a square into
$k = n - 2$ squares. Take one of these $n - 2$ squares and divide it into four identical smaller squares. This increases the number of subsquares by 3, producing a square broken up into $(n - 2) + 3 = n + 1$ squares. Thus if our claim is true for $n$, then it is also true for $n + 1$.

Consequently, by the Theorem on Mathematical Induction we may conclude that $(\diamondsuit)$ holds true.

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