Solution to Problems ♠-13

Problem A:  Let $a_n$ be the number written with $2^n$ nines. For example, $a_0 = 9$, $a_1 = 99$, $a_2 = 9999$. Let $b_n = \prod_{i=0}^{n} a_i$. Find the sum of the digits of $b_n$.

Answer: Answer: $9 \cdot 2^n$.

We prove this by induction on $n \in \mathbb{N}$. Let $P(n)$ be the statement that “the sum of the digits of $b_n$ is $9 \cdot 2^n$”. We are going to verify that $P(n)$ satisfies the assumptions of the Theorem on Mathematical Induction.

Basic Step: We have $b_0 = 9$, digit sum 9, and $b_1 = 891$, digit sum 18, so the result is true for $n = 0$ and $n = 1$.

Inductive Step: Let $n \in \mathbb{N}$ be arbitrary and let us assume $P(n-1)$ is true. Obviously $a_n < 10^{2^n}$, so

$$b_{n-1} < 10^{1+2+2^2+...+2^{n-1}} < 10^{2^n}.$$  

Now $b_n = b_{n-1} 10^{2^n} - b_{n-1}$. But $b_{n-1} < 10^{2^n}$, so

$$b_n = (b_{n-1} - 1) 10^{2^n} + (10^{2^n} - b_{n-1})$$

and the digit sum of $b_n$ is just the digit sum of $(b_{n-1} - 1) \cdot 10^{2^n}$ plus the digit sum of $10^{2^n} - b_{n-1}$.

Now $b_{n-1}$ is odd and so its last digit is non-zero, so the digit sum of $b_{n-1} - 1$ is one less than the digit sum of $b_{n-1}$, and hence is $9 \cdot 2^{n-1} - 1$. Multiplying by $10^{2^n}$ does not change the digit sum. $(10^{2^n} - 1) - b_{n-1}$ has $2^n$ digits, each 9 minus the corresponding digit of $b_{n-1}$, so its digit sum is $9 \cdot 2^n - 9 \cdot 2^{n-1}$. Since $b_{n-1}$ is odd, its last digit is not 0 and hence the last digit of $(10^{2^n} - 1) - b_{n-1}$ is not 9. So the digit sum of $10^{2^n} - b_{n-1}$ is $9 \cdot 2^n - 9 \cdot 2^{n-1} + 1$. Hence $b^n$ has digit sum

$$(9 \cdot 2^{n-1} - 1) + (9 \cdot 2^n - 9 \cdot 2^{n-1} + 1) = 9 \cdot 2^n.$$  

Consequently, by the Theorem on Mathematical Induction we may conclude that $(\forall n \in \mathbb{N})P(n)$, i.e., our assertion is true indeed.
Problem B: Show that for each natural number \( n \),

\[
\sum_{i=1}^{n} \frac{1}{i} \leq 1 + \ln(n).
\]

Answer: We show this by induction on \( n \in \mathbb{N} \). Let \( \Phi(n) \) be the statement \( \sum_{i=1}^{n} \frac{1}{i} \leq 1 + \ln(n) \). We will verify that \( \Phi(n) \) satisfies the assumptions of the Theorem on Mathematical Induction.

Basic Step: We note that \( \Phi(1) \) is the assertion that \( \sum_{i=1}^{1} \frac{1}{i} \leq 1 + \ln(1) \), but this is readily true.

Inductive Step: Let \( n \in \mathbb{N} \) be arbitrary and let us argue that the implication \( \Phi(n) \Rightarrow \Phi(n+1) \) holds true.

So suppose \( \Phi(n) \) is true, that is

\[
(\otimes)_n \sum_{i=1}^{n} \frac{1}{i} \leq 1 + \ln(n).
\]

Now note that

\[
\sum_{i=1}^{n+1} \frac{1}{i} = \frac{1}{n+1} + \sum_{i=1}^{n} \frac{1}{i} \leq (\otimes)_n \frac{1}{n+1} + 1 + \ln(n).
\]

Note that by the Lagrange’s Mean Value Theorem, for some \( x \in (n, n+1) \) we have

\[
\ln(n+1) - \ln(n) = \frac{1}{x} \geq \frac{1}{n+1},
\]

and hence

\[
\frac{1}{n+1} + 1 + \ln(n) \leq 1 + \ln(n+1).
\]

Consequently, under the assumption \( (\otimes)_n \), we have

\[
\sum_{i=1}^{n+1} \frac{1}{i} \leq 1 + \ln(n+1).
\]

By the Theorem on Mathematical Induction we may conclude that \( (\forall n \in \mathbb{N})\Phi(n) \), i.e., our claim is true indeed.

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