Solution to Problems ♠-10

Problem A: How many solutions in nonnegative integers are there to the equation

\[ x_1 + x_2 + x_3 + x_4 = 105 \]

Answer:

Every solution to our equation (in nonnegative integers) corresponds to placing 3 dividers in a sequence of 105 stones, thus creating 4 bins.

\[ x_1 \text{ stones} \big| x_2 \text{ stones} \big| x_4 \text{ stones} \big| x_4 \text{ stones} \]

And vice versa. (Note: if \( x_i = 0 \) then two dividers \( \big| \) are adjacent.)

Thus the problem asks in how many ways we may place 3 dividers between 105 objects. This amounts to choosing 3 spots out of 105 + 3 places, so it can be done in

\[
\binom{105 + 3}{3} = \frac{108!}{3! \cdot 105!} = \frac{108 \cdot 107 \cdot 106}{6} = 204156
\]

ways.

Correct solutions were received from:

(1) Brad Tuttle
Problem B:  Find the absolute maximum value of the function
\[ f(x) = \frac{1}{1 + |x|} + \frac{1}{1 + |x - 2|}, \quad x \in \mathbb{R}. \]

Answer:  We note that
\[ f(x) = \begin{cases} 
\frac{1}{1 + x} + \frac{1}{1 + (x - 2)} & \text{if } x < 0, \\
\frac{1}{1 + x} + \frac{1}{1 - (x - 2)} & \text{if } 0 \leq x < 2, \\
\frac{1}{1 + x} + \frac{1}{1 + (x - 2)} & \text{if } x \geq 2,
\end{cases} \]
and clearly the function \( f \) is continuous on its domain \( \mathbb{R} \) and differentiable on the intervals \( (-\infty, 0), (0, 2) \) and \( (2, \infty) \). Moreover,
\[ f'(x) = \begin{cases} 
\frac{-1}{(1-x)^2} + \frac{-1}{(3-x)^2} & \text{if } x < 0, \\
\frac{1}{(1+x)^2} + \frac{1}{(3-x)^2} & \text{if } 0 < x < 2, \\
\frac{-1}{(1+x)^2} - \frac{1}{(x-1)^2} & \text{if } x > 2.
\end{cases} \]
Plainly,
- if \( x < 0 \) then \( f'(x) > 0 \),
- if \( x > 2 \) then \( f'(x) < 0 \).
Also, for \( x \in (0, 2) \) we have
\[ f'(x) = \frac{-1}{(1 + x)^2} + \frac{1}{(3 - x)^2} = \frac{8(x - 1)}{(3 - x)^2 \cdot (x + 1)^2}. \]
Therefore,
- if \( 0 < x < 1 \) then \( f'(x) < 0 \),
- if \( 1 < x < 2 \) then \( f'(x) > 0 \), and
\[ f'(1) = 0. \]
Consequently, by the First Derivative Test, the local maxima of the function \( f \) are at \( x = 0 \) and \( x = 2 \), where takes the value \( \frac{4}{3} \). Therefore, \( \frac{4}{3} \) is the absolute maximum value of \( f \).

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