Project Title:
Combinatorics of finite vector spaces

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Description:
In a recent work, Rosłanowski and Rykov [1] showed that, consistently, there is a Borel set which has uncountably many pairwise very non-disjoint translations, but does not allow a perfect set of such translations. They gave an elaborate forcing arguments based on the celebrated work of Shelah [3]. The main new ingredients of the proof used somewhat strange combinatorial structure of finitely dimension vector spaces over the field $\mathbb{Z}_2$. For instance, the following lemma was crucial there.

Lemma [Rosłanowski and Rykov [1, Lemma 2.3(b)]] For $0 < \ell < \omega$ let ${}^{\ell}\{0,1\}$ denote the set of all 0–1 sequences of length ℓ. This set is equipped with coordinate-wise addition modulo 2, denoted $+$ and coordinate-wise multiplication by scalars from $\mathbb{Z}_2$. Let $\mathcal{B} \subseteq {}^{\ell}\{0,1\}$ be a linearly independent set of vectors (in $({}^{\ell}\{0,1\},+,\cdot)$ over $(\{0,1\},+_2,\cdot_2)$). If $\mathcal{A} \subseteq {}^{\ell}\{0,1\}$, $|\mathcal{A}| \geq 5$ and

$$\{a_1 + a_2 : a_1, a_2 \in \mathcal{A}\} \subseteq \{b_1 + b_2 : b_1, b_2 \in \mathcal{B}\},$$

then for a unique $x \in {}^{\ell}\{0,1\}$ we have $\{a + x : a \in \mathcal{A}\} \subseteq \mathcal{B}$.

It is not yet known if the consistency result of [1] can be proven for subsets of $\mathbb{R}$ or even for other product topological groups (like ${}^{\omega}4$ with coordinate-wise addition modulo 4). The first step in getting the respective proofs might be the analysis of the lemma stated above and investigations of its analogues in the respective vector spaces.

In this project we propose that the participating student engages in the following activities:

(A) Study the combinatorial arguments given in [1, Section 2] and also in [2, Section 3].
(B) Investigate the combinatorial properties of the finitely dimensional fields over $\mathbb{Z}_p$, try to decide what analogues of the lemma stated above are true for spaces of the form $({}^{\ell}\{0,1,\ldots,p-1\},+,\cdot)$ (over $\mathbb{Z}_p$).
(C) Analyze ways to extend the consistency of [1] onto other topological groups.

It is expected that the results of the work related to this project will include combinatorial background for the consistency result generalizing [1] and possibly a research paper.

Timetable:

January–March 2018	The student will read selected parts of research publications mentioned in this proposal.
April–May 2018:	The student will attempt to prove the expected properties of spaces similar to $({}^{\ell}\{0,1,\ldots,p-1\},+,\cdot)$. He will also attempt to discover new combinatorial properties of finite vector spaces.
May 2018:	Preparation of the final report and proposals for *NASA Nebraska Space Grant Fellowship* as well for *FUSE* grant.

References

[1] Andrzej Rosłanowski and Vyacheslav V. Rykov. Not so many non-disjoint translations. *Proceedings of the American Mathematical Society*, submitted. arxiv:1711.04058.

[2] Andrzej Roslanowski and Saharon Shelah. Small–large subgroups of the reals. *Mathematica Slovaca*, accepted. arxiv:1605.02261.

[3] Saharon Shelah. Borel sets with large squares. *Fundamenta Mathematicae*, 159:1–50, 1999. arxiv:math.LO/9802134.