Solution to Problems ♥–5

Problem A: For \( x > 0 \), show that
\[
\arctan(x) > \frac{3x}{1 + 2\sqrt{1 + x^2}}.
\]

Answer: Consider a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) given by formula
\[
f(x) = \arctan(x) - \frac{3x}{1 + 2\sqrt{1 + x^2}}.
\]
The function \( f \) is differentiable on \( \mathbb{R} \) and
\[
f'(x) = \frac{1}{1 + x^2} - \frac{3(1 + 2\sqrt{1 + x^2}) - 3x \cdot \frac{2x}{\sqrt{1 + x^2}}}{(1 + 2\sqrt{1 + x^2})^2} =
\]
\[
= \frac{(1 + 2\sqrt{1 + x^2})^2 - 3(1 + x^2) - 6(1 + x^2)^{3/2} + 6x^2 \sqrt{1 + x^2}}{(1 + x^2) \cdot (1 + 2\sqrt{1 + x^2})^2}
\]
\[
= \frac{1 + 4\sqrt{1 + x^2} + 4(1 + x^2) - 3(1 + x^2) - 6\sqrt{1 + x^2}((1 + x^2) - x^2)}{(1 + x^2) \cdot (1 + 2\sqrt{1 + x^2})^2}
\]
\[
= \frac{1 + 4\sqrt{1 + x^2} + (1 + x^2) - 6\sqrt{1 + x^2}}{(1 + x^2) \cdot (1 + 2\sqrt{1 + x^2})^2} = \frac{(1 - \sqrt{1 + x^2})^2}{(1 + x^2) \cdot (1 + 2\sqrt{1 + x^2})^2}.
\]
Now it should be clear that \( f'(x) > 0 \) for all \( x \in \mathbb{R} \) and the function \( f \) is strictly increasing on \( \mathbb{R} \). Since \( f(0) = 0 \), we may conclude that \( f(x) > 0 \) for all \( x > 0 \), i.e.,
\[
\arctan(x) > \frac{3x}{1 + 2\sqrt{1 + x^2}}
\]
for all \( x > 0 \).

Correct solution was received from:

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Problem B: Show that for all positive real numbers $x, y$ the following inequality holds:

$$x^y + y^x > 1.$$ 

Answer: The inequality is obvious if $x \geq 1$ or $y \geq 1$. So assume that $x, y \in (0, 1)$. In view of the symmetry, it is enough to consider the case where $y \leq x$. Let $t = y/x \in (0, 1]$. We have

$$x^y + y^x = x^{tx} + (tx)^x = (x^x)^t + t^x x^x.$$ 

Since the function $g : (0, \infty) \rightarrow (0, \infty) : x \mapsto x^x$ attains its minimum value $e^{-1/e} = a$ at $\frac{1}{e}$ and since $t^x \geq t$ (as $0 < t \leq 1$, $0 < x < 1$), we see that

$$x^y + y^x \geq a^t + ta.$$ 

Consider a function $G : [0, \infty) \rightarrow (0, \infty) : s \mapsto a^s + sa$. Clearly, $G$ is differentiable and

$$G'(s) = \ln(a)a^s + a = e^{-1/e}\left(-\frac{1}{e} + e^{\frac{s}{e} + \frac{1}{2}} + 1\right) = e^{-\frac{1}{e}}\left(-e^{\frac{s}{e}} + e\right) > 0$$ 

for all $s \geq 0$. Therefore $G$ is strictly increasing on $[0, \infty)$. Since $G(0) = 1$, we conclude that $G(t) > 1$ and hence

$$x^y + y^x \geq a^t + ta > 1.$$ 

Correct solution was received from:

(1) Brad Tuttle