Solution to Problems

Problem A: Let \((a_n)_{n=1}^{\infty}\) be a sequence of positive real numbers with all terms different from 1. Show that if \(\lim_{n \to \infty} a_n = 1\), then
\[
\lim_{n \to \infty} \frac{\ln(a_n)}{a_n - 1} = 1.
\]

Answer: First we note that

\((\star)\) for all \(x > -1\) we have \(\frac{x}{x+1} \leq \ln(1 + x) \leq x\).

[Why is \((\star)\) true? Consider functions \(f, g, h : (-1, \infty) \to \mathbb{R}\) defined by
\[f(x) = \frac{x}{x+1}, \quad g(x) = \ln(1 + x) \quad \text{and} \quad h(x) = x \quad \text{for } x > -1.\]
Clearly, \(f(0) = g(0) = h(0) = 0\) and for each \(x > -1\) we have
\[f'(x) = \frac{1}{(x+1)^2}, \quad g'(x) = \frac{1}{1+x}, \quad \text{and} \quad h'(x) = 1.\]

Hence, if \(x \geq 0\) then \(f'(x) \leq g'(x) \leq h'(x)\) and if \(-1 < x < 0\) then \(f'(x) > g'(x) > h'(x)\). Consequently, as \(f(0) = g(0) = h(0) = 0, f(x) \leq g(x) \leq h(x)\) for all \(x > -1\).]

Since \(\lim_{n \to \infty} a_n = 1 > 0\), for sufficiently large \(n\) we have \(a_n > 0\). Without loss of generality \(a_n > 0\) for all \(n\) and thus \(a_n - 1 > -1\) for all \(n\). It follows from \((\star)\) that, for all \(n,\)
\[
\frac{a_n - 1}{1 + (a_n - 1)} \leq \ln(1 + (a_n - 1)) = \ln(a_n) \leq a_n - 1.
\]

Hence, if \(a_n > 1\) then
\[
\frac{1}{a_n} \leq \frac{\ln(a_n)}{a_n - 1} \leq 1
\]
and therefore by the squeeze principle, the subsequence of \(\left(\frac{\ln(a_n)}{a_n - 1}\right)_{n=1}^{\infty}\) consisting of those terms for which \(a_n - 1 > 0\), if infinite, converges to 1. Similarly, if \(a_n < 1\) then
\[
\frac{1}{a_n} \geq \frac{\ln(a_n)}{a_n - 1} \geq 1,
\]
and therefore by the squeeze principle, the subsequence of \(\left(\frac{\ln(a_n)}{a_n - 1}\right)_{n=1}^{\infty}\) consisting of those terms for which \(a_n - 1 < 0\), if infinite, converges to 1. Consequently, \(\lim_{n \to \infty} \frac{\ln(a_n)}{a_n - 1} = 1\).

No correct solution were received
NOTE: my biggest apologies, but I made a typo in the Problem B, making the statement not true. The correct problem (with solution) are given below. —Andrzej Roslanowski.

Problem B: Let \((a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty\) be sequences of positive real numbers such that
\[
\lim_{n \to \infty} a_n^n = a > 0 \quad \text{and} \quad \lim_{n \to \infty} b_n^n = b > 0.
\]
Suppose that \(p, q > 0\) satisfy \(p + q = 1\). Prove that
\[
\lim_{n \to \infty} (pa_n + qb_n)^n = a^pb^q
\]

Answer: Note first that if \(\lim_{n \to \infty} a_n^n = a > 0\) then \(\lim_{n \to \infty} a_n = 1\). Assume now that the terms of the sequences \((a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty\) are different from 1. By Problem A, we have then

\[
\lim_{n \to \infty} \frac{n \ln(a_n)}{n(a_n - 1)} = 1. 
\]

But the assumption \(\lim_{n \to \infty} a_n^n = a > 0\) and the continuity of the logarithm function imply that \(\lim_{n \to \infty} n \ln(a_n) = \ln(a)\). Consequently, \((\star)\) gives us
\[
\lim_{n \to \infty} n(a_n - 1) = \lim_{n \to \infty} n \ln(a_n) = \ln(a).
\]

Note that the above equalities hold also for the subsequence consisting of the terms equal 1, if it is infinite. Therefore \(\lim_{n \to \infty} n(a_n - 1) = \ln(a)\) without the additional assumption on \(a_n\), and similarly also \(\lim_{n \to \infty} n(b_n - 1) = \ln(b)\).

Finally,
\[
\lim_{n \to \infty} n \ln(pa_n + qb_n) = \lim_{n \to \infty} n[p(a_n - 1) + q(b_n - 1)] = \ln(a^pb^q).
\]

(Above we use again Problem A for \(c_n = pa_n + qb_n\).)

Correct solution was received from:

(1) Cody Anderson

POW 11B: ♥
Problem ♣ C: Find the limit of the sequence \((a_n)_{n=1}^{\infty}\), where
\[a_n = \left(1 + \frac{1}{n^2}\right) \cdot \left(1 + \frac{2}{n^2}\right) \cdot \ldots \cdot \left(1 + \frac{n}{n^2}\right), \quad \text{for } n = 1, 2, 3, \ldots\]

Answer: First we note that
(♣) for all \(x > 0\) we have \(x - \frac{x^2}{2} < \ln(1 + x) < x\).
[Why is (♣) true? Consider functions \(f, g, h : (-1, \infty) \rightarrow \mathbb{R}\) defined by
\[f(x) = x - \frac{x^2}{2}, \quad g(x) = \ln(1 + x) \quad \text{and} \quad h(x) = x \quad \text{for } x > -1.\]
Clearly, \(f(0) = g(0) = h(0) = 0\) and for each \(x > 0\) we have
\[f'(x) = 1 - x < g'(x) = \frac{1}{1+x} < h'(x) = 1.\]
Consequently, as \(f(x) < g(x) < h(x)\) for all \(x > 0\).]
Let
\[b_n = \ln(a_n) = \sum_{k=1}^{n} \ln\left(1 + \frac{k}{n^2}\right).\]
By (♣) we have
\[\frac{k}{n^2} - \frac{k^2}{2n^4} < \ln\left(1 + \frac{k}{n^2}\right) < \frac{k}{n^2}.\]
It is well known that
\[\sum_{k=1}^{n} k = \frac{n(n + 1)}{2} \quad \text{and} \quad \sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6}\]
(easy inductive proofs). Therefore
\[\frac{n(n + 1)}{2n^2} - \frac{n(n + 1)(2n + 1)}{12n^4} < b_n < \frac{n(n + 1)}{2n^2},\]
and hence \(\lim_{n \to \infty} b_n = \frac{1}{2}\). Consequently,
\[\lim_{n \to \infty} a_n = \sqrt{e}.\]

Correct solution was received from:
(1) Cody Anderson