PROBLEM SOLVING CONTEST

UNO, October 30, 2009

Read the following Instructions:

This is a test consisting of 5 problems. Each problem is worth 10 points, assigned on the significant steps you are able to take in writing the solution. To help the graders assign partial credit, please carefully show your work on each problem. Your work will be graded by two graders independently of each other. Your final score on each problem will be the average of the scores entered by these graders. Your total score will be the sum of these 5 average scores. The participants with the top 3 total scores will be designated as winners of the I-st, II-d, and III-d prizes, respectively. Their prizes will be mailed C/O their mathematics instructor, so besides your name and school affiliation, please do not forget to write the name of your math instructor. There is a traveling trophy for this contest. The school with the best team score from their top three participants will receive the trophy. The trophy will be sent to the winning team for display until next year when this contest will be organized again.

You have exactly one hour and 30 minutes to work on the problems. Rather than just guessing answers, please show work and explain your statements on each problem. Good luck!

Problem 1 *Suppose all 6-digit numbers in base 10 are printed in numerical order with leading zeros included. So the first number printed is* 000000 *and the last number printed is* 999999*. How many times is a leading* 0 *printed ?*

Solve the problem mathematically (show work) and give a numerical final answer.

Problem proposed by John Konvalina.

Solution

The number of leading 0's can be counted by columns. In the 1's column there is 1 leading 0, in the 10's column there are 10 leading 0's, and so on up to the $100,00$'s column. So the total number of leading 0's is

$$1 + 10 + 100 + 1000 + 10000 + 100000 = 111,111.$$

Problem 2 *Solve the following equation*

$$3^{3x}5^{x-1} = 3^{x+1}5^{3x}.$$

Problem proposed by Andrew Tew.

Solution

Apply the natural logarithm to the entire equation and apply the rules of operations with logarithms to get the following equivalent expressions:

$$\ln(3^{3x}) + \ln(5^{x-1}) = \ln(3^{x+1}) + \ln(5^{3x}) \Leftrightarrow$$

$$3x \cdot \ln(3) + (x-1) \cdot \ln(5) = (x+1) \cdot \ln(3) + 3x \cdot \ln(5) \Leftrightarrow$$

$$2x \cdot \ln(3) - 2x \cdot \ln(5) = \ln(3) + \ln(5) \Leftrightarrow$$

$$2x \cdot (\ln(3) - \ln(5)) = \ln(3) + \ln(5).$$

Solving for x we obtain

$$x = \frac{\ln(3) + \ln(5)}{2 \cdot (\ln(3) - \ln(5))} = -2.65066.$$

Problem 3 *One fall afternoon you go to Vala's pumpkin patch with four of your friends. Each one of you picks out a pumpkin to take home. When it is time to weigh the five pumpkins to determine their cost, you and your friends decide to weigh them two at a time in all ten sets of two.*

The weights that you record are $32, 34, 35, 36, 37, 38, 39, 40, 42,$ *and* 43.

Assuming that the pumpkin weights are integers, and that no two pumpkins have the same weight, how much does each individual pumpkin weigh?

Problem proposed by Julia Warnke.

Solution

Label the pumpkins A, B, C, D, E. Since no two pumpkins have the same weight, we can order them as follows: $A < B < C < D < E$.

We know that $A + B$ will have the smallest value and $A + C$ will have the next smallest value. We also know that $D + E$ will have the largest value and $C + E$ will have the next largest value.

We also know that each pumpkin was weighed four times according to the pairings. Thus the total of the recorded weights, $282 = 4(A+B+C+D+E)$.

The equations that we know are the following:

$$94 = (A + B + C + D + E)$$

$$32 = A + B$$

$$34 = A + C$$

$$42 = C + E$$

$$43 = D + E$$

From there we can easily find the values of the 5 variables.

$$D = (A + B + C + D + E) - (A + B) - (C + E) = 94 - 32 - 42 = 20$$

$$C = (A + B + C + D + E) - (A + B) - (D + E) = 94 - 32 - 43 = 19$$

$$E = (D + E) - D = 43 - 20 = 23$$

$$A = (A + C) - C = 34 - 19 = 15$$

$$B = (A + B) - A = 32 - 15 = 17$$

So

$$A = 15 < B = 17 < C = 19 < D = 20 < E = 23.$$

Problem 4 *Let N be the natural number obtained by concatenating all the natural numbers in increasing order starting with* 1. *That is, let*

$$N = 1234567891011121314151617181920 21...$$

Find the 20008*-th digit of N.*

Problem proposed by Dora Matache.

Solution

For writing the numbers from 1 to 9 we need $9 \times 1 = 9$ digits.

For writing the numbers from 10 to 99 we need $90 \times 2 = 180$ digits.

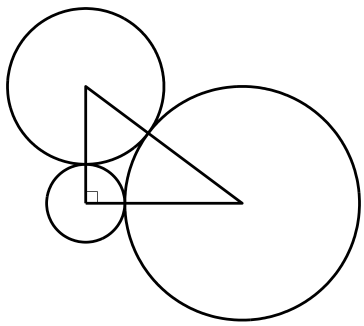
For writing the numbers from 100 to 999 we need $900 \times 3 = 2700$ digits.

For writing the numbers from 1000 to 9999 we need $9000 \times 4 = 36000$ digits.

Since for writing all the numbers of at most three digits we need $9+180+2700 = 2889$ digits, and for writing all the numbers of at most four digits we need $9 + 180 + 2700 + 36000 = 38889$ digits, it means that the 20008-th digit of N belongs to a natural number of four digits. Therefore we need $20008 - 2889 = 17119$ digits to write the four-digit numbers in order to reach the 20008-th digit of N.

Now, $17119 = 4279 \times 4 + 3$, so the 20008-th digit of N is the third digit of the 4280-th four digit number written in N. Since this one is exactly $1000 + 4280 - 1 = 5279$, its third digit is 7. Thus the 20008-th digit of N is equal to 7.

Problem 5 *The triangle in the figure is a 3-4-5 right triangle, that is the lengths of its sides are* $3, 4, 5$. *The vertices of the triangle are the centers of three mutually tangent circles. What is the sum of the areas of the three circles?*



Problem proposed by Renate Keimig.

Solution

Let the radius of the small circle be a. Then the radius of the second circle is $(3 - a)$, and the radius of the large circle is $(4 - a)$. Since the hypotenuse of the triangle is 5, we know that $(3 - a) + (4 - a) = 5$. Therefore, $a = 1$.

Substituting a into the equations for the radii, for the smallest circle, $r = 1$, and $A = \pi r^2 = \pi$.

For the second circle, $r = 2$, and $A = \pi r^2 = 4\pi$.

For the largest circle, $r = 3$, and $A = \pi r^2 = 9\pi$.

The sum of the areas is $\pi + 4\pi + 9\pi = 14\pi$.