PROBLEM SOLVING CONTEST
MATH AWARENESS MONTH 2006

UNO, April 21, 2006

Read the following Instructions:

This is a test consisting of 5 problems. Each problem is worth 10 points, assigned on the significant steps you are able to take in writing the solution. To help the graders assign partial credit, please carefully show your work on each problem. Your work will be graded by two graders independently of each other. Your final score on each problem will be the average of the scores entered by these graders. Your total score will be the sum of these 5 average scores. The participants with the top 3 total scores will be designated as winners of the I-st, II-d, and III-d prizes, respectively. Their prizes will be mailed C/O their mathematics instructor, so besides your name and school affiliation, please do not forget to write the name of your math instructor. There is a travelling trophy for this contest. The school with the best team score from their top three participants will receive the trophy. The trophy will be sent to the winning team for display until next April when this contest will be organized again.

You have exactly one hour and 30 minutes to work on the problems. Rather than just guessing answers, please show work and explain your statements on each problem. Good luck!

Note: Please DO NOT write on the back of the pages, only on the same side with the text of the problems! If you need more space we will be happy to give you some paper on which you should write only on one side.

Problem 1 *How many ways can the letters in the word MATHEMATICS be arranged so that all the vowels are adjacent, that is, next to each other? Here is an example: MTHMTCSAEAI. Find a mathematical solution of this problem.*

Solution

There are 7 consonants: M, T, H, M, T, C, S. There are 4 vowels: A, E, A, I.

First, think of the vowels as a block and permute with the consonants to get $8!/(2!2!)$. Then, permute the vowels within the block in $4!/2!$ ways. So the final answer is the product: $8!/(2!2!) \cdot 4!/2!$. Or $120,960$.

Problem 2 *Show that the following expression is constant.*

$$\sqrt{\sin^2 x + \csc^2 x + \cos^2 x + \sec^2 x - (\tan^2 x + \cot^2 x)}$$

Solution

We use the following facts: $\sin^2 x + \cos^2 x = 1$, $\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sin x}$, $\tan x = \frac{\sin x}{\cos x}$, and $\cot x = \frac{\cos x}{\sin x}$. Replacing all these in the expression under the radical we get

$$1 + \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} - \left(\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\sin^2 x}\right)$$

Performing the operations we get

$$1 + \frac{1}{\sin^2 x \cos^2 x} - \frac{\sin^4 x + \cos^4 x}{\sin^2 x \cos^2 x} = 1 - \frac{\sin^4 x + \cos^4 x - 1}{\sin^2 x \cos^2 x}$$

Now, we know that in general $a^2 + b^2 = (a+b)^2 - 2ab$. We apply this to the sum $\sin^4 x + \cos^4 x$ to obtain

$$1 - \frac{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x - 1}{\sin^2 x \cos^2 x} = 1 - \frac{1 - 2\sin^2 x \cos^2 x - 1}{\sin^2 x \cos^2 x} = 1 + 2 = 3$$

Therefore the original expression is equal to $\sqrt{3}$ and is constant.

Problem 3 *A triangle with integral sides has a perimeter of 8. What is the area of the triangle?*

Solution

Let the sides of the triangle be represented by x, y, z, where $x \geq y \geq z \geq 0$. Since x must be at least $1/3$ the perimeter and less than half the perimeter and an integer, we know that $x = 3$. Thus we have $y + z = 5$. Since y and z must also be integers, we must have $y = 3$ and $z = 2$. Thus we have an isosceles triangle of sides $3, 3$, and 2. The area of such a triangle is $2\sqrt{2}$ (the altitude to the side of length 2 is $\sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$ so that the area is $1 \cdot 2\sqrt{2} = 2\sqrt{2}$).

Problem 4 *Solve the following equation:*

$$\sqrt{x\sqrt{x\sqrt{x\cdots}}} = \sqrt{2006}$$

Solution

Let $y = \sqrt{x\sqrt{x\sqrt{x\cdots}}}$, then $y = \sqrt{2006}$. On the other hand, all radicals without the first term will also be equal to y i.e. $\sqrt{xy} = \sqrt{2006}$. Then $xy = 2006$ and $x\sqrt{2006} = 2006$, thus $x = \sqrt{2006} \approx 44.788$.

Problem 5 *A path consisting of 100 square stones is to be painted. The length of a side of the first stone is 1 foot; the length of a side of the second stone is 2 feet; and so on, until the length of a side of the final stone is 100 feet. If one gallon of paint covers 101 square feet, how many gallons of paint will be necessary to paint the entire path?*

Solution

Since the nth square has a side of length n, its area will be n^2. Thus the entire path will have area

$$1^1 + 2^2 + 3^2 + \cdots + 100^2$$

Use the fact that $1^1 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ to obtain that the total square footage

$$\frac{100 \times 101 \times 201}{6} = 3350 \times 101$$

Since 101 square feet can be painted with one gallon, we divide the total square footage by 101 to get the number of gallons os paint needed, that is

3350