

Vladimir Ufimtsev. Problem R-7.

First, since the interval we are looking for is  $[a, b]$ , then  $a \leq b$ . From the first requirement that  $[a, b]$  contains no integers, we know  $b \leq a + 1$ , and thus  $|b - a - 1| = a + 1 - b$ . From the second requirement we have the following equation:

$$a + 1 - b = a^2 + b^2 + \frac{a}{2} - 2b + \frac{21}{16}$$

Or:

$$-a^2 + \frac{a}{2} + (b - b^2 - \frac{5}{16}) = 0$$

Using the quadratic formula:

$$a = \frac{1}{4} \pm \frac{\sqrt{\frac{1}{4} + 4(b - b^2 - \frac{5}{16})}}{2}$$

Note that the term under the radical can be expressed as:

$$\frac{1}{4} + 4(b - b^2 - \frac{5}{16}) = -(4b^2 - 4b + 1) = -(2b - 1)^2$$

Note that  $-(2b - 1)^2 \leq 0$ . Since  $a \in \mathbb{R}$  we must have:  $2b - 1 = 0$  or  $b = \frac{1}{2}$ . This implies that  $a = \frac{1}{4}$ .

The interval that satisfies the requirements is  $[\frac{1}{4}, \frac{1}{2}]$  and is unique.