Vladimir Ufimtsev. Problem H-7.

Assume:
$\mathbf{P}: (|z_1 + 2z_2| = 3|z_3|) \wedge (|z_1 + 3z_3| = 2|z_2|) \wedge (|2z_2 + 3z_3| = |z_1|)$
$\mathbf{Q}: z_1 + 2z_2 + 3z_3 = 0$

To show that propositions $\mathbf{P}$ and $\mathbf{Q}$ are equivalent we must show that (1) $\mathbf{Q} \to \mathbf{P}$ and (2) $\mathbf{P} \to \mathbf{Q}$.

(1) $\mathbf{Q} \to \mathbf{P}$. From the equation: $z_1 + 2z_2 + 3z_3 = 0$ we deduce that:

$z_1 + 2z_2 = -3z_3$
$z_1 + 3z_3 = -2z_2$
$2z_2 + 3z_3 = -z_1$

By taking the Euclidean norm of both sides of the 3 equations we obtain:

$|z_1 + 2z_2| = |-3z_3| = 3|z_3|$
$|z_1 + 3z_3| = |-2z_2| = 2|z_2|$
$|2z_2 + 3z_3| = |-z_1| = |z_1|$

Therefore $\mathbf{Q} \to \mathbf{P}$

(2) $\mathbf{P} \to \mathbf{Q}$. Let $z_1 = a_1 + b_1 i$,$z_2 = a_2 + b_2 i$, $z_3 = a_3 + b_3 i$. Noting the fact that $|a + bi| = \sqrt{a^2 + b^2}$ and squaring each of the 3 equations in $\mathbf{P}$ we obtain:

$(a_1^2 + 4a_2^2 + 4a_1a_2) + (b_1^2 + 4b_2^2 + 4b_1b_2) = 9(a_3^2 + b_3^2)$
$(a_1^2 + 9a_3^2 + 6a_1a_3) + (b_1^2 + 9b_3^2 + 6b_1b_3) = 4(a_2^2 + b_2^2)$
$(4a_2^2 + 9a_3^2 + 12a_2a_3) + (4b_2^2 + 9b_3^2 + 12b_2b_3) = (a_1^2 + b_1^2)$

By summing the 3 equations we obtain:

$$(a_1^2+4a_2^2+9a_3^2+4a_1a_2+6a_1a_3+12a_2a_3)+(b_1^2+4b_2^2+9b_3^2+4b_1b_2+6b_1b_3+12b_2b_3) = 0$$

Factoring the above equation:

$$(a_1 + 2a_2 + 3a_3)^2 + (b_1 + 2b_2 + 3b_3)^2 = 0$$

Which means that:

$$|z_1 + 2z_2 + 3z_3| = 0$$

By the definition of norm, this occurs only when $z_1 + 2z_2 + 3z_3 = 0$. Therefore $\mathbf{P} \to \mathbf{Q}$.

Thus, propositions $\mathbf{P}$ and $\mathbf{Q}$ are equivalent.