The Choquet Integral with Respect to Fuzzy-Valued Set Functions

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Abstract

The Choquet integral with respect to real-valued nonadditive set functions, such as signed efficiency measures, has been used in information fusion and data mining as an aggregation tool successfully. However, in some real-world problems, the values of the involved set function may be not crisp but fuzzy. So the Choquet integral needs to be generalized such that it can be taken with respect to interval-valued signed efficiency measure and, more generally, fuzzy-valued signed efficiency measure. A calculation formula for such kind of Choquet integral is developed when the universal set, such as the set of attributes in a database, is finite and the fuzzy-valued set function takes trapezoidal fuzzy numbers, involving triangular fuzzy numbers as their special cases. Some properties of the Choquet integrals with respect to fuzzy-valued efficiency measures are also discussed.

Keywords. Nonadditive set functions, the Choquet integral, fuzzification.

1. Introduction

As an aggregation tool, nonlinear integrals, such as the Choquet integral with respect to nonadditive set functions have been widely applied in information fusion and data mining. Up to now, in the existing nonlinear multiregression models and nonlinear classifiers, the nonlinear integrals are taken with respect to real-valued set functions. However, in many real problems, allowing set functions to assume fuzzy numbers seems more reasonable. Hence, this paper tries to establish the Choquet integral with respect to fuzzy-valued
signed efficiency measures. Using such Choquet integrals in multiregression is one of the approaches of fuzzy information retrieval.

To develop the calculation formula of the Choquet integral with respect to fuzzy-valued signed efficiency measures, there is a problem when integrand takes both positive and negative values since the subtraction is not the inverse operation of the addition for fuzzy numbers. Decomposition theorem of fuzzy sets and extension principle are used to cope with the problem.

The paper is organized as follows. In Section 2, the relevant mathematical concepts are introduced. In Section 3, we show the interval-valued signed efficiency measure on $\mathcal{P}(X)$. Section 4 presents the fuzzy-valued signed efficiency measure on $\mathcal{P}(X)$. In Section 5 we discuss the properties of such kind of Choquet integral with respect to fuzzy-valued signed efficiency measures. Finally, brief conclusions are given in Section 6.

2. Relevant mathematical concepts

Let $X$ be a nonempty set, called the universal set. We also let $\mathcal{F}$ be a $\sigma$-algebra, consisting of subsets of $X$. If $X$ is finite, usually we take $\mathcal{P}(X)$ as $\sigma$-algebra $\mathcal{F}$. $(X, \mathcal{F})$ is called a measurable space.

**Definition 1** [5]. Set function $\mu : \mathcal{P}(X) \rightarrow (-\infty, \infty)$ is called a monotone measure ( is also called fuzzy measure ) if

1. $\mu(\emptyset) = 0$ ;
2. $\mu(E) \geq 0 \quad \forall E \in \mathcal{F}$ ;
3. $\mu(E) \leq \mu(F)$ if $E, F \in \mathcal{F}$ and $E \subseteq F$.

If $\mu(X) = 1$, monotone measure $\mu$ is called normalized.
Definition 2 [5]. Set function $\mu : \mathcal{F}(X) \rightarrow (-\infty, \infty)$ is called an efficiency measure (is also called generalized measure) if it satisfies the condition (1) and (2) given in Definition 1.

Any monotone measure is a special case of efficiency measures.

Definition 3 [5]. Set function $\mu : \mathcal{F}(X) \rightarrow (-\infty, \infty)$ is called a signed efficiency measure (is also called signed generalized measure) if it satisfies only condition (1) given in Definition 1.

Let $f : X \rightarrow (-\infty, \infty)$ be a measurable function with respect to $\mathcal{F}$. When $\mu$ is a monotone measure, the Choquet integral of $f$ with respect to $\mu$ can be defined as follows.

Definition 4 [5]. Let $\mu$ be a monotone measure on $(X, \mathcal{F})$. The Choquet integral of $F$ with respect to $\mu$, denoted by $(C) \int f d\mu$, is defined as

$$(C) \int f d\mu = \int_{-\infty}^{0} [\mu(F_{\alpha}) - \mu(X)] d\alpha + \int_{0}^{\infty} \mu(F_{\alpha}) d\alpha$$

(1)

if not both Riemann integrals in (1) are infinite, where $F_{\alpha} = \{x \mid f(x) \geq \alpha\}$ for $\alpha \in (-\infty, \infty)$ and is called the $\alpha$-level set of $f$.

The $\alpha$-level set $F_{\alpha}$ is nonincreasing with respect to $\alpha$. Since $\mu$ is nondecreasing, $\mu(F_{\alpha})$ is a nonincreasing function of $\alpha$, such that both Riemann’s integral in Definition 4 are well defined.

As a special case, when $f$ is nonnegative, formula (1) is reduced to be
\[
(C) \int f \, d\mu = \int_0^\infty \mu(F_\alpha) \, d\alpha.
\]  

(2)

In most cases, formula (1) is also feasible for the Choquet integral with respect to efficient measures.

**Definition 5** [6]. Let \( \mu \) be a signed efficiency measure on \( \mathcal{F} \). A pair of two efficiency measures, \( \nu^+ \) and \( \nu^- \), satisfying \( \mu(A) = \nu^+(A) - \nu^-(A) \) \( \forall A \in \mathcal{F} \) (simply, we write \( \mu = \nu^+ - \nu^- \)) is called a nonnegative decomposition of \( \mu \).

We may omit word “nonnegative” in the above definition if there is no confusion, and simply call it “a decomposition”. For a given signed efficiency measure, there are infinitely many decompositions. Among them, there is a smallest one.

**Definition 6** [6]. The smallest decomposition of signed efficiency measure \( \mu \) is the decomposition, \( \mu^+ \) and \( \mu^- \), such that \( \mu^+ \leq \nu^+ \) and \( \mu^- \leq \nu^- \) for any decomposition, \( \nu^+ \) and \( \nu^- \), of \( \mu \).

The smallest decomposition of \( \mu \) is unique. It can be expressed as

\[
\mu^+(A) = \begin{cases} 
\mu(A) & \text{if } \mu(A) \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
\mu^-(A) = \begin{cases} 
-\mu(A) & \text{if } \mu(A) \leq 0 \\
0 & \text{otherwise}
\end{cases}
\]

for any \( A \in \mathcal{F} \). \( \mu^+ \) and \( \mu^- \) are called the positive part and the negative part of \( \mu \) respectively.
**Definition 7** [5]. Let \( \mu \) be a signed efficiency measure and \( f \) be a real-valued measurable function on \((X, \mathcal{F})\).

The Choquet integral of \( f \) with respect to \( \mu \) is defined as

\[
(C) \int f \, d\mu = (C) \int f \, d\mu_1 - (C) \int f \, d\mu_2
\]

if not both Choquet integral in formula (4) are infinite, where \( \mu_1 \) and \( \mu_2 \) are efficiency measures and form the smallest decomposition of \( \mu \).

In any database, the number of attributes is always finite, that is, \( X \) is a finite set. In this case, there is a simple formula for calculating the value of \((C) \int f \, d\mu\) once \( f \) and \( \mu \) are given. First, the values of function \( f \), \( \{f(x_1), f(x_2), \ldots, f(x_n)\} \), are rearranged into a nondecreasing order as,

\[
f(x_1^*) \leq f(x_2^*) \leq \ldots \leq f(x_n^*)
\]

where \((x_1^*, x_2^*, \ldots, x_n^*)\) is a permutation of \((x_1, x_2, \ldots, x_n)\). Then the Choquet integral of \( f \) with respect to \( \mu \) can be calculated by

\[
(C) \int f \, d\mu = \sum_{i=1}^{n} [f(x_i^*) - f(x_{i-1}^*)] \mu(\{x_i^*, x_{i+1}^*, \ldots, x_n^*\})
\]

with a convention \( f(x_0^*) = 0 \).

**Definition 8** [2]. Let \( A \) be a fuzzy subset of universal set \( X \). For any \( \alpha \in [0,1] \), crisp set \( \{x \mid m_d(x) \geq \alpha, x \in X\} \) is called the \( \alpha \)–cut of \( A \), denoted by \( A_{\alpha} \); while crisp set \( \{x \mid m_d(x) > \alpha, x \in X\} \) is called the strong \( \alpha \)–cut of \( A \), denoted by \( A_{\alpha+} \). It is clear that \( A_0 = X, A_{\alpha} = \text{supp} \, A \), and \( A_{\alpha+} = \emptyset \).
**Definition 9** [2]. For any crisp set $E$ and any real number $\alpha \in [0,1]$, $\alpha E$ is the fuzzy set having membership function

$$m_{\alpha E}(x) = \begin{cases} \alpha & \text{if } x \in E \\ 0 & \text{if } x \not\in E \end{cases} \forall x \in X.$$  

Let $A$ be a fuzzy subset of universal set $X$.

**Theorem 1** (Decomposition theorem I) [2]. $A = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha = \bigcup_{\alpha \in [0,1]} aA_\alpha$.

**Theorem 2** (Decomposition theorem II) [2]. $A = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha^+ = \bigcup_{\alpha \in [0,1]} aA_\alpha^+.$

**Definition 10** [2]. Set $\{\alpha | m_A(x) = \alpha \text{ for some } x \in X\}$ is called the level-value set of fuzzy set $A$, denoted by $L_A$.

**Theorem 2** (Decomposition theorem III) [2]. $A = \bigcup_{\alpha \in L_A} \alpha A_\alpha$.

**Theorem 3** (Extension principle) [2]. Let $X_1$, $X_2$, ..., $X_n$, and $Y$ be nonempty crisp sets, $U = X_1 \times X_2 \times \cdots \times X_n$ be the product set of $X_1$, $X_2$, ..., and $X_n$, and $f: U \to Y$ be a mapping from $U$ to $Y$. Then mapping $f$ can be extended to be $f: \mathcal{F}(X_1) \times \mathcal{F}(X_2) \times \cdots \times \mathcal{F}(X_n) \to \mathcal{F}(Y)$ as follows: for any given $n$ fuzzy sets $A_i \in \mathcal{F}(X_i)$, $i = 1, 2, \ldots, n$, fuzzy set $B = f(A_1, A_2, \ldots, A_n) \in \mathcal{F}(Y)$ has membership function

$$m_B(y) = \sup_{x_1, x_2, \ldots, x_n | y = f(x_1, x_2, \ldots, x_n)} \min[m_{A_1}(x_1), m_{A_2}(x_2), \ldots, m_{A_n}(x_n)]$$

with convention
\[ \sup_{x \in [0, 1]} x = 0. \]

As a special case, if \(*\) is a binary operator on universal set \(X\), that is, \(*: X \times X \to X\), then, by the extension principle, we can obtain a binary operator on \(\mathcal{F}(X)\): for any \(A, B \in \mathcal{F}(X)\).

\[ m_{A \ast B}(z) = \sup_{x \ast y = z} [m_A(x) \wedge m_B(y)] \quad \forall z \in X. \]

By using classical extension principle, the common operations of real numbers can be extended to be operations for interval numbers. Any real number \(a\) can be regarded as an interval number \([a, a]\). The set of all interval numbers is denoted by \(I\) while the set of all nonnegative interval numbers is denoted by \(I_+\).

To quantify fuzzy concepts, some types of fuzzy subsets of \(R = (-\infty, +\infty)\) are used. Fuzzy numbers are a most common type of fuzzy subsets of \(R\) for this purpose.

**Definition 11** [2]. Any closed interval \([a, b]\), where \(-\infty < a \leq b < \infty\) is called an interval number. Any interval number \([a, b]\) satisfying \(a \geq 0\) is called a nonnegative interval number.

**Definition 12** [2]. A fuzzy number, denoted by a capital letter with a wave such as \(\tilde{A}\), is a fuzzy subset of \(R\) with membership function \(m: R \to [0, 1]\) satisfying the following conditions:

(FN1) \(\tilde{A}_\alpha\), the \(\alpha\)-cut of \(\tilde{A}\), is a closed interval for any \(\alpha \in (0, 1]\);

(FN2) \(\tilde{A}_0\) is bounded.
Condition (FN1) implies the convexity of \( \tilde{A} \), i.e., any fuzzy number is a convex fuzzy subset of \( R \). For any \( \alpha \in (0, 1] \), the \( \alpha \)-cut of a fuzzy number is an interval number. The set of all fuzzy numbers is denoted by \( \mathcal{N}_F \).

**Definition 13** [2]. Fuzzy number \( \tilde{A} \) is said to be nonnegative if its \( \tilde{A}_\alpha = [a, b] \) with \( a \geq 0 \) for all \( \alpha \in (0, 1] \).

**Theorem 4** [2]. Condition (FN1) is equivalent to the following conditions:

(FN1.1) there exists at least one real number \( a_0 \) such that \( m(a_0) = 1 \);

(FN1.2) \( m(t) \) is nondecreasing on \( (-\infty, a_0] \) and nonincreasing on \( [a_0, \infty) \);

(FN1.3) \( m(t) \) is upper semi-continuous, or say, \( m(t) \) is right-continuous on \( (-\infty, a_0) \) and is left-continuous on \( (a_0, +\infty) \).

For any fuzzy number with membership function \( m(t) \), there exists a closed interval \([a_b, a_c]\) such that

\[
m(t) = \begin{cases} 
1 & t \in [a_b, a_c] \\
l(t) & t \in (-\infty, a_b) \\
r(t) & t \in (a_c, +\infty)
\end{cases}
\]

where \( 0 \leq l(t) < 1 \), called the left branch of \( m(t) \), is nondecreasing and \( 0 \leq r(t) < 1 \), called the right branch of \( m(t) \), is nonincreasing.

**Definition 14** [2]. A rectangular fuzzy number is a fuzzy number with membership function having form as
A fuzzy number is rectangular iff the left branch and right branch of its membership function are zero. It is identified with the corresponding vector \([a_i, a_r]\) and is an interval number essentially. Any crisp real number \(a\) can be regarded as a special rectangular fuzzy number with \(a_i = a_r = a\).

**Definition 15** [2]. A triangular fuzzy number is a fuzzy number with membership function

\[
m(t) = \begin{cases} 
1 & \text{if } t \in [a_b, a_c] \\
0 & \text{otherwise} 
\end{cases},
\]

where \(a_i, a_r \in R\) with \(a_i \leq a_r\).

A triangular fuzzy number is identified with corresponding vector \([a_i, a_o, a_r]\). Any crisp real number \(a\) can be regarded as a special triangular fuzzy number with \(a_i = a_o = a_r = a\).

**Definition 16** [5]. A trapezoidal fuzzy number is a fuzzy number with membership function

\[
m(t) = \begin{cases} 
1 & \text{if } t = a_o \\
\frac{t - a_i}{a_o - a_i} & \text{if } t \in [a_i, a_o] \\
\frac{t - a_r}{a_o - a_r} & \text{if } t \in (a_o, a_r) \\
0 & \text{otherwise} 
\end{cases},
\]

where \(a_i, a_o, a_r \in R\) with \(a_i \leq a_o \leq a_r\).
\[
m(t) = \begin{cases} 
1 & \text{if } t \in [a_b, a_c] \\
\frac{t-a_i}{a_b-a_i} & \text{if } t \in [a_i, a_b) \\
\frac{t-a_c}{a_c-a_r} & \text{if } t \in (a_c, a_r] \\
0 & \text{otherwise}
\end{cases}
\]

where \( a_i, a_b, a_c, a_r \in R \) with \( a_i \leq a_b \leq a_c \leq a_r \).

A trapezoidal fuzzy number is identified with the corresponding vector \([a_i, a_b, a_c, a_r]\). Fig. 1 shows the membership function of a trapezoidal fuzzy number, denoted by \( E \), and its \( \alpha \)-cut. Any rectangular fuzzy number \([a, b]\) can be regarded as a special trapezoidal fuzzy number with \( a_i = a_b \) and \( a_c = a_r \). Similarly, any triangular fuzzy number \([a, a_0, a_r]\) can be regarded as a special trapezoidal fuzzy number with \( a_b = a_c = a_0 \). Of course, any crisp real number \( a \) can be regarded as a special trapezoidal fuzzy number with \( a_i = a_b = a_c = a_r = a \). Thus, our discussion and models can be applicable to database involving even both crisp and fuzzy data.

Both the left branch and the right branch of the membership function of a trapezoidal fuzzy number are piecewise linear.

Notation \( \alpha[a, b] \) should be understood as a fuzzy set possessing membership function

\[
m_{\alpha[a, b]}(t) = \begin{cases} 
\alpha & \text{if } t \in [a, b] \\
0 & \text{otherwise}
\end{cases}
\]

Now we restrict in using trapezoidal fuzzy numbers for the fuzzy-valued signed efficiency measure in the Choquet integral.
Let $A$ and $B$ be two trapezoidal fuzzy members with membership functions

$$m_A(t) = \begin{cases} 
\frac{t-a_1}{c_1-a_1} & \text{if } t \in [a_1, c_1) \\
1 & \text{if } t \in [c_1, d_1] \\
\frac{b_1-t}{b_1-d_1} & \text{if } t \in (d_1, b_1] \\
0 & \text{otherwise}
\end{cases}$$

and

$$m_B(t) = \begin{cases} 
\frac{t-a_2}{c_2-a_2} & \text{if } t \in [a_2, c_2) \\
1 & \text{if } t \in [c_2, d_2] \\
\frac{b_2-t}{b_2-d_2} & \text{if } t \in (d_2, b_2] \\
0 & \text{otherwise}
\end{cases}$$

respectively. Then,
\[ A_\alpha = [\alpha(c_1 - a_1) + a_1, \ b_1 - \alpha(b_1 - d_1)], \]
\[ B_\alpha = [\alpha(c_2 - a_2) + a_2, \ b_2 - \alpha(b_2 - d_2)]. \]

Since
\[ (A + B)_\alpha = A_\alpha + B_\alpha = [\alpha(c_1 + c_2 - a_1 - a_2) + a_1 + a_2, \ b_1 + b_2 - \alpha(b_1 + b_2 - d_1 - d_2)], \]
we obtain
\[
m_{A+B}(t) = \begin{cases} 
\frac{t - (a_1 + a_2)}{(c_1 + c_2) - (a_1 + a_2)} & \text{if } t \in [a_1 + a_2, c_1 + c_2) \\
1 & \text{if } t \in [c_1 + c_2, d_1 + d_2] \\
\frac{(b_1 + b_2) - t}{(b_1 + b_2) - (d_1 + d_2)} & \text{if } t \in (d_1 + d_2, b_1 + b_2] \\
0 & \text{otherwise}
\end{cases}.
\]

This means that the sum of any two trapezoidal fuzzy numbers is still a trapezoidal fuzzy number. It is evident that a trapezoidal fuzzy number multiplied by a constant is still a trapezoidal fuzzy number. Thus, any linear combination of trapezoidal fuzzy numbers is still a trapezoidal fuzzy number that can be determined by the end points and the top points of the original trapezoidal fuzzy numbers.

**Definition 17** [5]. Set function \( \mu : \mathfrak{F}(X) \rightarrow \mathcal{N}_I \), denoted by \( \overline{\mu} \), is called an interval-valued signed efficiency measure, if \( \overline{\mu}(\emptyset) = 0 \).

**Definition 18** [5]. Set function \( \mu : \mathfrak{F}(X) \rightarrow \mathcal{N}_F \), denoted by \( \tilde{\mu} \), is called a fuzzy-valued signed efficiency measure, if \( \tilde{\mu}(\emptyset) = 0 \).
**Definition 19** [5]. For any fuzzy-valued signed efficiency measure $\tilde{\mu}$ on $(X, \mathcal{F})$ and any $\alpha \in (0,1]$, the interval-valued signed efficiency measure $\tilde{\mu}_\alpha (A) = [\tilde{\mu}(A)]^\alpha$ for very $A \in \mathcal{F}$ is called the $\alpha$–cut of $\tilde{\mu}$.

3. **The Choquet Integral with Interval-valued signed efficiency measure on $\mathcal{P}(X)$**

When $X$ is a finite nonempty set, according to Theorem 1 (Decomposability), the calculation of the Choquet integral with respect to an interval-valued signed efficiency measure is realizable. Let $X = \{x_1, x_2, ..., x_n\}$, $\mu$ be an interval-valued signed efficiency measure on $\mathcal{P}(X)$, and $f$ be a given real-valued function on $X$. Similar to Definition 7, rearrange $\{x_1, x_2, ..., x_n\}$ into $X = \{x_1^*, x_2^*, ..., x_n^*\}$ such that $f(x_1^*) \leq f(x_2^*) \leq ... \leq f(x_n^*)$ and let $f(x_0^*) = 0$. Then the Choquet integral of $f$ with respect to $\mu$ can be calculated by

$$(C)\int f \, d\mu = \sum_{i=1}^{n} [f(x_i^*) - f(x_{i-1}^*)] \cdot \mu(\{x_i^*, x_{i+1}^*, ..., x_n^*\}).$$

Denote

$$\mu(\{x_1^*, x_{i+1}^*, ..., x_n^*\}) = [a_i, b_i],$$

and

$$f(x_i^*) - f(x_{i-1}^*) = \Delta_i, \text{ for } i = 1, 2, ..., n,$$

we have

$$(C)\int f \, d\mu = \sum_{i=1}^{n} \Delta_i \cdot [a_i, b_i].$$

Noting $\Delta_i \geq 0$ and $\Delta_1 = f(x_1^*) = \min_{1 \leq i \leq n} f(x_i)$, we obtain
Example 1 Let $X = \{x_1, x_2, x_3\}$, $\mu$ be an interval-valued signed efficiency measure on $\mathcal{P}(X)$ defined as

$$
\mu(A) = \begin{cases} 
[0, 0] & \text{if } A = \emptyset \\
[1, 2] & \text{if } A = \{x_1\} \\
[-1, 1] & \text{if } A = \{x_2\} \\
[3, 5] & \text{if } A = \{x_1, x_2\} \\
[-2, 3] & \text{if } A = \{x_3\} \\
[1, 4] & \text{if } A = \{x_1, x_3\} \\
[-5, -2] & \text{if } A = \{x_2, x_3\} \\
[5, 6] & \text{if } A = X 
\end{cases}
$$

and $f$ be a real-valued function on $X$ expressed by

$$
f = \begin{cases} 
-0.3 & \text{if } x = x_1 \\
0.2 & \text{if } x = x_2, \\
-0.5 & \text{if } x = x_3 
\end{cases}
$$

then, $x_1^* = x_3, x_2^* = x_1$ and $x_3^* = x_2$.

Furthermore, $\Delta_1 = -0.5 < 0$, $\Delta_2 = -0.3 - (-0.5) = 0.2$, $\Delta_3 = 0.2 - (-0.3) = 0.5$, $[a_1, b_1] = [5, 6]$, $[a_2, b_2] = [3, 5]$ and $[a_3, b_3] = [-1, -1]$. Thus, by using (4),

$$
(C) \int f \, d\mu = \begin{cases} 
\left[ \sum_{i=2}^{n} \Delta_i a_i + \Delta_i b_i, \sum_{i=2}^{n} \Delta_i b_i + \Delta_i a_i \right] & \text{if } \Delta_i \geq 0 \\
\left[ \sum_{i=2}^{n} \Delta_i a_i + \Delta_i b_i, \sum_{i=2}^{n} \Delta_i b_i + \Delta_i a_i \right] & \text{if } \Delta_i \leq 0 
\end{cases}
$$

(4)
(C) \[ \int f d\mu = [\Delta_2 a_2 + \Delta_3 a_3 + \Delta_1 b_1, \Delta_2 b_2 + \Delta_3 b_3 + \Delta_1 a_1] \]
\[= [0.2 \times 3 + 0.5 \times (-1) + (-0.5) \times 6, 0.2 \times 5 + 0.5 \times 1 + (-0.5) \times 5] \]
\[= [-2.9, -1]. \]

4. The Choquet integral with fuzzy-valued signed efficiency measure on \( \mathcal{A}(X) \)

Let \( \tilde{\mu} : \mathcal{F} \rightarrow \mathcal{N}_F \), be a fuzzy-valued signed efficiency measure and \( f \) be a measurable function on measurable space \((X, \mathcal{F})\). Since the \( \alpha \) – cut of any given fuzzy number is a closed interval for every \( \alpha \in (0, 1] \) and the Choquet integral with respect to an interval-valued signed efficiency measure can be expressed in a linear form of the values of the interval-valued signed efficiency measure, the Choquet integral with respect to a signed fuzzy-valued efficiency measure can be expressed by using the composition theorem of fuzzy sets. Thus, the Choquet integral of \( f \) with respect to \( \tilde{\mu} \) is

\[ \text{(C)} \int f d\tilde{\mu} = \bigcup_{\alpha \in [0, 1]} \alpha \cdot \text{(C)} \int f d\mu_{\alpha}, \]

where \( \mu_{\alpha} \) is the \( \alpha \) – cut of \( \tilde{\mu} \), that is, the \( \alpha \) – cut of fuzzy number value and it is an interval value.

When Let \( X = \{x_1, x_2, ..., x_n\} \), that is, \( X \) is finite, we rearrange \( \{x_1, x_2, ..., x_n\} \) into \( \{x_1^*, x_2^*, ..., x_n^*\} \) such that \( f(x_1^*) \leq f(x_2^*) \leq ... \leq f(x_n^*) \) and let \( f(x_0^*) = 0 \). Then the Choquet integral of \( f \) with respect to \( \tilde{\mu} \) can be expressed as

\[ \text{(C)} \int f d\tilde{\mu} = \begin{cases} \bigcup_{\alpha \in [0, 1]} \alpha \left[ \sum_{i=2}^{n} \Delta_i a_i(\alpha) + \Delta_i a_i(\alpha), \sum_{i=2}^{n} \Delta_i b_i(\alpha) + \Delta_i b_i(\alpha) \right] & \text{if } \Delta_1 \geq 0 \\
\bigcup_{\alpha \in [0, 1]} \alpha \left[ \sum_{i=2}^{n} \Delta_i a_i(\alpha) + \Delta_i b_i(\alpha), \sum_{i=2}^{n} \Delta_i b_i(\alpha) + \Delta_i a_i(\alpha) \right] & \text{if } \Delta_1 \leq 0 
\end{cases} \]
where \( a_i(\alpha) \) and \( b_i(\alpha) \) are the end points of interval

\[ [a_i(\alpha), b_i(\alpha)] = \bar{\mu}_\alpha \left( \{x_{i^*}, x_{i+1}^*, \ldots, x_n^*\} \right), \]

and \( \Delta_i = f(x_i^*) - f(x_{i-1}^*) \) for \( i = 1, 2, \ldots, n \),

with \( f(x_0^*) = 0 \). This makes the calculation of the Choquet integral with respect to trapezoidal fuzzy-valued signed efficiency measure rather easy. The formula is

\[
(C) \int f d \tilde{\mu} = \begin{cases} 
\left[ \sum_{i=2}^{n} \Delta_i a_i + \Delta_i a_1, \sum_{i=1}^{n} \Delta_i d_i, \sum_{i=1}^{n} \Delta_i b_i + \Delta_i b_1 \right] & \text{if } \Delta_i \geq 0 \\
\left[ \sum_{i=2}^{n} \Delta_i a_i + \Delta_i b_1, \sum_{i=1}^{n} \Delta_i c_i, \sum_{i=1}^{n} \Delta_i d_i, \sum_{i=1}^{n} \Delta_i b_i + \Delta_i a_1 \right] & \text{if } \Delta_i \leq 0 
\end{cases},
\]

where \( a_i, b_i, \) and \( c_i, d_i \) are the end points and top points of trapezoidal fuzzy number \([a_i, c_i, d_i, b_i] = \tilde{\mu}(\{x_{i^*}, x_{i+1}^*, \ldots, x_n^*\})\) and \( \Delta_i = f(x_i^*) - f(x_{i-1}^*) \) for \( i = 1, 2, \ldots, n \), respectively.

**Example 2** Let \( X = \{x_1, x_2, x_3\} \), \( \tilde{\mu} \) be a fuzzy-valued signed efficiency measure on \( \mathcal{P}(X) \) defined as

\[
\tilde{\mu}(A) = \begin{cases} 
[0, 0, 0, 0] & \text{if } A = \emptyset \\
[1, 1.5, 3.5, 2] & \text{if } A = \{x_1\} \\
[-1, 0, -1, 1] & \text{if } A = \{x_2\} \\
[3, 4.5, 1, 5] & \text{if } A = \{x_1, x_2\} \\
[-2, -5, 1, 3] & \text{if } A = \{x_3\} \\
[1, 2, 3, 4] & \text{if } A = \{x_1, x_3\} \\
[-5, 1, -3, -2] & \text{if } A = \{x_2, x_3\} \\
[5, 6, 3, 6] & \text{if } A = X
\end{cases},
\]
\[ f(x) = \begin{cases} 
-0.3 & \text{if } x = x_1 \\
0.2 & \text{if } x = x_2 \\
-0.5 & \text{if } x = x_3 
\end{cases} \]

then, \( x_i^* = x_3, x_2^* = x_1 \) and \( x_3^* = x_2 \). Thus, as shown in (6), \( \int f d\bar{\mu} \) is still a trapezoidal fuzzy number, whose end points are just the end points of the resulting interval in Example 1, that is, –2.9 and –1. As for the top point of \( (C) \int f d\bar{\mu} \) is the Choquet integral of \( f \) with respect to signed efficiency measure, whose value at any set in \( \mathcal{P}(X) \) is just the top points of the value (a trapezoidal fuzzy number) of \( \bar{\mu} \) at the same set, that is,

\[
\begin{align*}
(C) \int f d\mu_c &= (-0.5) \times 6 + 0.2 \times 4.5 + 0.5 \times 0 = -2.7, \\
(C) \int f d\mu_d &= (-0.5) \times 3 + 0.2 \times 1 + 0.5 \times (-1) = -1.8,
\end{align*}
\]

where

\[
\mu_c(A) = \begin{cases} 
0 & \text{if } A = \emptyset \\
1.5 & \text{if } A = \{x_1\} \\
0 & \text{if } A = \{x_2\} \\
4.5 & \text{if } A = \{x_1, x_2\} \\
-5 & \text{if } A = \{x_3\} \\
2 & \text{if } A = \{x_1, x_3\} \\
1 & \text{if } A = \{x_2, x_3\} \\
6 & \text{if } A = X
\end{cases}
\]
Consequently,

\[
(C) \int f d \tilde{\mu} = \{-2.9, -2.7, -1.8, -1\},
\]

A trapezoidal fuzzy number with end points \(-2.9\) and \(-1\), top point \(-2.7\) and \(-0.8\), whose membership function is

\[
m(t) = \begin{cases}
    \frac{t+2.9}{0.2} & \text{if } t \in [-2.9, -2.7) \\
    1 & \text{if } t \in [-2.7, -1.8] \\
    \frac{-1-t}{0.8} & \text{if } t \in (-1.8, -1] \\
    0 & \text{otherwise}
\end{cases}
\]

5. Properties of the Choquet integral with respect to fuzzy-valued signed efficiency measures

The choquet integral of nonnegative measurable function \(f\) with respect to a fuzzy-valued signed efficiency measure \(\tilde{\mu}\) has following basic properties, where we
assume that all involved functions and sets are measurable:

(CIP1) \( (C) \int fd \tilde{\mu} \geq 0; \)

(CIP2) \( (C) \int c \cdot fd \tilde{\mu} = c \cdot (C) \int fd \tilde{\mu} \) for any nonnegative constant \( c. \)

These properties can be obtained from the definition directly. However, \( (C) \int (f + g)d \tilde{\mu} = (C) \int fd \tilde{\mu} + (C) \int gd \tilde{\mu} \) may not be true. This can be seen from the following example.

**Example 3.** Let \( X = \{a, b\} \), \( \mathcal{F} = \mathcal{P}(X) \), and

\[
\tilde{\mu}(E) = \begin{cases} 
[0, 0, 0, 0] & \text{if } E = \emptyset \\
[1, 1, 1, 1] & \text{if otherwise}
\end{cases}
\]

In this case, any function on \( X \) is measurable. Considering two functions,

\[
f(x) = \begin{cases} 
0 & \text{if } x = a \\
1 & \text{if } x = b
\end{cases}
\]

and

\[
g(x) = \begin{cases} 
0 & \text{if } x = b \\
1 & \text{if } x = a
\end{cases}
\]

we have

\[
(C) \int fd \tilde{\mu} = [0, 0, 0, 0]
\]

and

\[
(C) \int gd \tilde{\mu} = [1, 1, 1, 1].
\]

Since \( f + g = 1 \), a constant function on \( X \), we obtain
(C) \( (f + g) d \tilde{\mu} = (C) \int 1 d \tilde{\mu} = 1 \cdot \tilde{\mu}(X) = [1, 1, 1, 1] \).

Thus,
\[
(C) \int (f + g) d \tilde{\mu} \neq (C) \int fd \tilde{\mu} + (C) \int gd \tilde{\mu}.
\]

(CIP3) When \( \tilde{\mu} \) is a nonnegative fuzzy-valued signed efficiency measure,
\[
(C) \int fd \tilde{\mu} \leq (C) \int gd \tilde{\mu} \quad \text{if} \quad f \leq g.
\]

(CIP4) \( (C) \int (f + c) d \tilde{\mu} = (C) \int fd \tilde{\mu} + c \cdot \tilde{\mu}(X) \) for any constant \( c \).

6. Conclusions

The paper presents that the Choquet integral of a real-valued measurable function can be generalized for interval-valued signed efficiency measures, and further more for fuzzy-valued signed efficiency measures. When the fuzzy values are restricted to trapezoidal fuzzy numbers, the calculation formula of such kind of Choquet integral is clearly presented.

For nonnegative measurable functions, the Choquet integral with respect to a fuzzy-valued signed efficiency measure preserves some properties of the Choquet integral with respect to a signed efficiency measure.
References


