PROBLEM SOLVING CONTEST
MATH AWARENESS MONTH 2008

UNO, April 18, 2008

Read the following Instructions:

This is a test consisting of 5 problems. Each problem is worth 10 points, assigned on the significant steps you are able to take in writing the solution. To help the graders assign partial credit, please carefully show your work on each problem. Your work will be graded by two graders independently of each other. Your final score on each problem will be the average of the scores entered by these graders. Your total score will be the sum of these 5 average scores. The participants with the top 3 total scores will be designated as winners of the I-st, II-d, and III-d prizes, respectively. Their prizes will be mailed C/O their mathematics instructor, so besides your name and school affiliation, please do not forget to write the name of your math instructor. There is a travelling trophy for this contest. The school with the best team score from their top three participants will receive the trophy. The trophy will be sent to the winning team for display until next April when this contest will be organized again.

You have exactly one hour and 30 minutes to work on the problems. Rather than just guessing answers, please show work and explain your statements on each problem. Good luck!

Note: Please DO NOT write on the back of the pages, only on the same side with the text of the problems! If you need more space we will be happy to give you some paper on which you should write only on one side.

Problem 1 *Burger Queen advertises that you can have your hamburger with or without any or all of the following toppings: catsup, cheese, lettuce, mayonnaise, mushrooms, mustard, onion, pickle, or tomato. Determine the number of different hamburger orders possible.*

Solution

There are 9 toppings to choose from and you either pick the topping or you don't. So the answer is 2^9. Or, you can sum the binomial coefficients of row 9 of Pascal's Triangle and get the same answer.

Problem 2 *(For this problem see Figures 1 and 2 on the last page)*

The large cube in Figure 1 is made up of 13 double cubes like the smaller one above (two cubes glued together: one with 6 red sides and one with 6 white sides), plus a single cube. The large cube looks just like this no matter which edge you are looking at (or in other words the other three faces not shown, continue this pattern). What color is the single cube? What are all of the positions that the single cube could occupy? How do you know that you're right?

Solution

Since by counting we find that there are 14 red showing and 12 white showing, we can see that the cube in the inside of the large cube must be white. If it were red, we would have 15 reds and 12 whites. Grouping these by red/white double cubes, we'd have 12 double cubes and 3 red single cubes left, which is impossible since we needed 13 double cubes and only 1 single cube. So, the middle must be white.

Then we would have 14 reds and 13 whites, leading us to see that we would have 13 double cubes and a single left over red cube. Where could that cube be? Well, it can't be in the center of the large cube, but it could be anywhere else that red cubes exist on the outside.

Here is how you can see that. Make the two lower 3x3x1 layers by stacking two cubes vertically, as shown in Figure 2.

Then what are we left? We need to use 4 more double cubes and 1 single red. Here are two possible combinations. The double cubes are bordered with bold lines. You could lay the double cubes around the edge with the single one in the middle of this layer. Or you could lay the single cube on the corner and then lay the 4 double cubes as shown appropriately.

So we've shown that the single cube can be anywhere on the top layer, but since this shape is symmetric, I could arbitrarily pick whichever level I am going to call the "top layer" so I could put the single cube on any of the possible red positions.

Problem 3 *Find the sum of the following infinite sequence:*

$$30,\ \frac{60}{\sqrt{5}+\sqrt{3}},\ 30\left(\sqrt{5}-\sqrt{3}\right)^2,\ \frac{60(\sqrt{5}-\sqrt{3})^2}{\sqrt{5}+\sqrt{3}},\ 30\left(\sqrt{5}-\sqrt{3}\right)^4,\ \frac{60(\sqrt{5}-\sqrt{3})^4}{\sqrt{5}+\sqrt{3}}\cdots$$

Solution

Since $(\sqrt{5}+\sqrt{3})\cdot(\sqrt{5}-\sqrt{3}) = 5-3 = 2$, we could multiply all the even terms of the series by $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ and the denominators become equal to 2. Thus the series can be written as follows:

$$30,\ 30\left(\sqrt{5}-\sqrt{3}\right),\ 30\left(\sqrt{5}-\sqrt{3}\right)^2,\ 30\left(\sqrt{5}-\sqrt{3}\right)^3,\ 30\left(\sqrt{5}-\sqrt{3}\right)^4,\cdots$$

So we have to compute the sum

$$S = 30+30\left(\sqrt{5}-\sqrt{3}\right)+30\left(\sqrt{5}-\sqrt{3}\right)^2+30\left(\sqrt{5}-\sqrt{3}\right)^3+30\left(\sqrt{5}-\sqrt{3}\right)^4+\cdots$$

$$= 30\cdot\left[1+\left(\sqrt{5}-\sqrt{3}\right)+\left(\sqrt{5}-\sqrt{3}\right)^2+\left(\sqrt{5}-\sqrt{3}\right)^3+\left(\sqrt{5}-\sqrt{3}\right)^4+\cdots\right]$$

which is a geometric series with ratio $0 < \sqrt{5}-\sqrt{3} < 1$, so the sum of the series is obtained as $\frac{1}{1-(\sqrt{5}-\sqrt{3})}$. Thus the sum is

$$S = \frac{30}{1-\sqrt{5}+\sqrt{3}}$$

Problem 4 *A number is called "miraculous" if it is natural and equals the sum of the squares of two of its distinct divisors.*

♣ *Give an example of a "miraculous" number. (Here is an example of a natural number that is NOT "miraculous":* $125 = 10^2 + 5^2$*, but* 10 *does not divide* 125*.)*

♠ *Show that there are at least 2008 "miraculous" numbers.*

Solution

♣ An example of a "miraculous" number is $20 = 4^2 + 2^2$ and 2 and 4 are two distinct divisors of 20.

♠ For any natural number k we have that $20k^2 = (4^2 + 2^2)k^2 = (4k)^2 + (2k)^2$. Clearly $2k$ and $4k$ are two distinct divisors of $20k^2$. Thus we have an infinity of "miraculous" numbers and therefore at least 2008.

Problem 5 *Find the area of the triangle* ABC *that has coordinates:*

$$A = (2,3), B = (8, 6\sqrt{3}+3), C = (2+4\sqrt{3}, 7)$$

Solution

Observe that it is easy to compute the length of each side of the triangle. Namely, we use the formula for the distance between two points of the plane (x_1, y_1) and (x_2, y_2) given by $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Thus the distance between A and B is $a = \sqrt{(2-8)^2 + (3 - 6\sqrt{3} - 3)^2} = 12$, the distance between B and C is $b = \sqrt{(8 - 2 - 4\sqrt{3})^2 + (6\sqrt{3} + 3 - 7)^2} = 4\sqrt{13 - 6\sqrt{3}}$, and the distance between C and A is $c = \sqrt{(2 - 2 - 4\sqrt{3})^2 + (3-7)^2} = 8$.

Now, Heron's formula for the area of a triangle given its sides a, b, c is $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a+b+c)/2$, that is half the perimeter. Replacing a, b, c we obtain $s = 10 + 2\sqrt{13 - 6\sqrt{3}}$ and consequently $A = 24$.

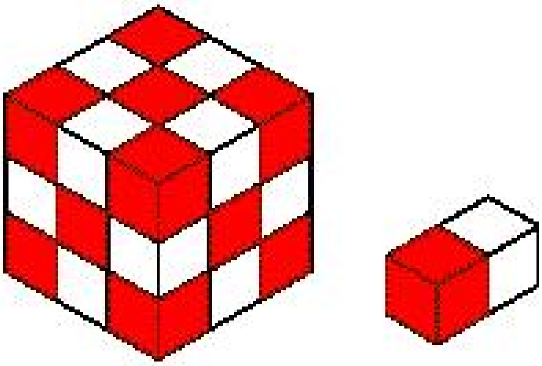

Figure 1:

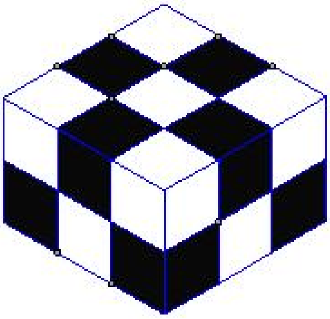

Figure 2: