PROBLEM SOLVING CONTEST
MATH AWARENESS MONTH 2007

UNO, April 27, 2007

Problem 1 *How may even integers between 4000 and 7000 have four different digits?*

Solution First look at the integers that have a 4 or a 6 for the thousands digit.

There are 2 ways to choose the thousands digit. There are 4 ways to choose the units digit (can be one of 0, 2, 4, 6, 8, but not the same as the thousands digit). There are 8 ways to choose the hundreds digit (can be any digit other than the same as the thousands or units digit). There are 7 ways to choose the tens digit (can be any digit other than the same as the thousand, units, or hundreds digit).

Thus there are 2x4x8x7 or 448 integers that have a 4 or a 6 for the thousand digit.

Now look at the integers that have a 5 for the thousands digit.

There is 1 way to choose the thousands digit. There are 5 ways to choose the units digit (can be any one of 0, 2, 4, 6, 8). As above, there are 8 ways to choose the hundreds digit and 7 ways to choose the tens digit.

Thus there are 1x5x8x7 or 280 integers that have a 5 for the thousand digit.

Thus there are 448+280 = 728 in all.

Problem 2 *Find all scalene triangles (that is the sides have unequal length) having all sides of integral lengths, and perimeter less than 13.*

Solution

Let the largest side be c. Then $a + b + c \leq 12$. But $c < a + b$ therefore $2c < 12$ or $c < 6$. Now try integral combinations such that the triangle is scalene. Since c is the largest side and $c < a + b$, it cannot be less than 4. There are 3 combinations.

$$c = 5, a = 4, b = 3; \qquad c = 5, a = 4, b = 2; \qquad c = 4, a = 3, b = 2$$

Problem 3 *A person has a hours at his disposal. How many miles may he ride in a car traveling b miles per hour and yet have time to return on foot walking c miles per hour?*

Solution

Denote by x the distance travelled with the car (or on foot on the way back). Since $time = \frac{distance}{speed}$, and the total time has to be a hours, we get

$$\frac{x}{b} + \frac{x}{c} = a$$

Solving for x we get $x = \frac{abc}{b+c}$.

Problem 4 *If*

$$(1 + \frac{1}{x^2})(1 + \frac{1}{x^4})(1 + \frac{1}{x^8})(1 + \frac{1}{x^{16}})(1 + \frac{1}{x^{32}}) = \frac{x^m - 1}{x^m - x^{m-2}}$$

then $m = ?$

Solution

Observe that the left hand side of the expression can be written as

$$\frac{x^2+1}{x^2}\frac{x^4+1}{x^4}\frac{x^8+1}{x^8}\frac{x^{16}+1}{x^{16}}\frac{x^{32}+1}{x^{32}}$$

We use the fact that $(a-b)(a+b) = a^2 - b^2$ to observe that if we multiply the expression by $x^2 - 1$ we get the following sequence of computations

$$(x^2-1)\frac{x^2+1}{x^2}\frac{x^4+1}{x^4}\frac{x^8+1}{x^8}\frac{x^{16}+1}{x^{16}}\frac{x^{32}+1}{x^{32}} =$$

$$= \frac{x^4-1}{x^2}\frac{x^4+1}{x^4}\frac{x^8+1}{x^8}\frac{x^{16}+1}{x^{16}}\frac{x^{32}+1}{x^{32}} =$$

$$= \frac{x^8-1}{x^6}\frac{x^8+1}{x^8}\frac{x^{16}+1}{x^{16}}\frac{x^{32}+1}{x^{32}} =$$

$$= \frac{x^{16}-1}{x^{14}} \frac{x^{16}+1}{x^{16}} \frac{x^{32}+1}{x^{32}} =$$
$$= \frac{x^{32}-1}{x^{30}} \frac{x^{32}+1}{x^{32}} =$$
$$\frac{x^{64}-1}{x^{62}}$$

Thus the left hand side of our original expression can be written as

$$\frac{x^{64}-1}{x^{62}(x^2-1)} = \frac{x^{64}-1}{x^{64}-x^{62}}$$

Thus $m = 64$.

Problem 5 *A company of soldiers is placed in rows forming an equilateral triangle: in the first row there is one soldier; in the second row there are two soldiers; in the third row there are three soldiers etc. If* 669 *soldiers are added to the company, they can now be placed in a square formation whose side is* 8 *soldiers less than the side of the equilateral triangle of the initial company. Find the initial number of soldiers.*

Solution

First observe that if we denote by x the side of the equilateral triangle formed with the initial company, then the initial number of soldiers can be expressed as:

$$S = 1 + 2 + 3 + \cdots + x = \frac{x(x+1)}{2}$$

However, when adding 669 to this number, we can form a square whose side is $x - 8$. Then the new number of soldiers is $(x-8)^2$. Thus we can write

$$S + 669 = (x-8)^2 \Rightarrow \frac{x(x+8)}{2} + 669 = (x-8)^2$$

This is a quadratic equation that can be solved for x. Observe that it can be written as $x^2 - 33x - 1210 = 0$ whose only positive solution is $x = 55$. Then $S = \frac{55 \cdot 56}{2} = 1540$ soldiers in the initial company.