PROBLEM SOLVING CONTEST
MATH AWARENESS MONTH 2005

UNO, April 15, 2005

Read the following Instructions:

This is a test consisting of 5 problems. Each problem is worth 10 points, assigned on the significant steps you are able to take in writing the solution. To help the graders assign partial credit, please carefully show your work on each problem. Your work will be graded by two graders independently of each other. Your final score on each problem will be the average of the scores entered by these graders. Your total score will be the sum of these 5 average scores. The participants with the top 3 total scores will be designated as winners of the I-st, II-d, and III-d prizes, respectively. Their prizes will be mailed C/O their mathematics instructor, so besides your name and school affiliation, please do not forget to write the name of your math instructor. There is a travelling trophy for this contest. The school with the best team score from their top three participants will receive the trophy. The trophy will be sent to the winning team for display until next April when this contest will be organized again.

You have exactly one hour and 30 minutes to work on the problems. Rather than just guessing answers, please show work and explain your statements on each problem. Good luck!

Note: Please DO NOT write on the back of the pages, only on the same side with the text of the problems! If you need more space we will be happy to give you some paper on which you should write only on one side.

Problem 1 *Which are the last* 3 *digits of* 2005^{2005}*? Give reasons for your answer.*

Solution:

Observe that given the equality $2005^{2005} = (2000+5)^{2005}$, this number is representable as a number ending in 3 zeros plus 5^{2005}. Note that the last three digits of 2005^{2005} are identical to the last three digits of 5^{2005}. By examining the successive powers of 5: 5, 25, 125, 625, 3125, 15625 ... note that powers of odd index, larger that 1 always end in 125. Since 2005 is odd, the last three digits of 2005^{2005} are 1, 2, and 5.

Problem 2 *There are 100 different books on a shelf and three different boxes on the floor. Choose any 20 books (order not important) and place them in the first box. Next, choose any 30 books from the shelf and place them in the second box. Finally, place the remaining books on the shelf into the third box. How many different ways can this be done?*

Solution:

There are $\binom{100}{20} = \frac{100!}{20!80!}$ ways to choose 20 books out of 100 (order not important). There are $\binom{80}{30} = \frac{80!}{30!50!}$ ways of choosing the next 30 books out of the remaining 80 books on the shelf. Thus the total number of ways is

$$\binom{100}{20}\binom{80}{30} = \frac{100!}{20!30!50!}$$

Problem 3 *The altitude drawn to the base of an isosceles triangle is* 8*, and the perimeter is* 32*. Find the area of the triangle.*

Solution:

Denote each side by a and the base by $2b$. Observe that $2a+2b=32$ and $a+b=16$. Apply the identity $a^2-b^2=(a-b)(a+b)$ to get $(a-b)\times 16 = 64$, since $a^2-b^2=8^2$ by the Pythagorean theorem. Solve the system: $a+b=16$, $a-b=4$ to get $b=6$. Find the area $Area = \frac{2\times 6\times 8}{2} = 48$.

Problem 4 *Three married couples inherit altogether* $\$10,000$. *The wives together receive* $\$3,960$, *distributed in such a way that Jean receives* $\$100$ *more than Kate, and May* $\$100$ *more than Jean. Albert Smith gets fifty per cent more than his wife, Henry March gets as much as his wife, while Thomas Hughes inherits twice as much as his wife. What are the three girls' last names?*

Solution:

If Kate got K dollars, then Jean got $K+100$ and May $K+200$. Observe that the total is then $3K+300 = 3960$ so $K = 1,220$ dollars. No matter who's married to whom, the men got $K + 1.5K + 2K = 4.5K = 5,940$ dollars, plus $10,000 - 3,960 - 5,940 = 550$. Remark that one of the numbers 0, 100, and 200, (which are the sums Kate, Jean respectively May got over $1,220$ dollars) should remain unchanged, another one should be doubled, and the remaining one should be multiplied by 1.5. The sum of the resulting numbers should be 550. The only way this can function is $0 + 1.5 \times 100 + 2 \times 200 = 550$, (since we increased numbers as much as possible, so anything else would return a smaller sum). So Kate got 0 dollars over $1,220$ and her husband got no extra money, hence Kate is married to Henry March. The extra 100 dollars went to Jean and her husband is the one who got the extra 150, so Jean is married to Albert Smith. Finally May must be Tomas Hughes's wife. The final answer: **Kate March, Jean Smith, and May Hughes.**

Problem 5 *The coefficient of* x^2 *in the expansion of the product* $(x+a)(x+b)(x+c)$ *is* 0. *The coefficient of* x *in the expansion of the product* $(x-a)(x+b)(x+c)$ *is* 0. *The coefficient of* x *in the first expansion is equal to the coefficient of* x^2 *in the second expansion. What must* a *be?*

Solution:

The coefficient of x^2 in the first expansion is $a+b+c$ and so $a+b+c = 0$. The coefficient of x in the second expansion is $bc-ac-ab$ and so $bc-ac-ab = 0$. The coefficient of x in the first expansion is $ab+ac+bc$ and the coefficient of x^2 in the second expansion is $c - a + b$. Thus we have a third equation $ab+ac+bc = c-a+b$. So we obtain the following system of three equations with three unknowns:

$$a + b + c = 0, \qquad bc - ac - ab = 0, \qquad ab + ac + bc = c - a + b$$

From the first equation we get $b + c = -a$. We replace this in the other two equations to get

$$bc + a^2 = 0, \qquad bc - a^2 = -2a$$

Subtracting the two equations, we obtain

$$2a^2 = 2a$$

which implies that a can be either 0 or 1.