PROBLEM SOLVING CONTEST
MATH AWARENESS MONTH 2004

UNO, April 1, 2004

Note: Any alternative correct solution receives full credit.

Problem 1 *Find the number*

$$N = \log_2 3 \times \log_3 4 \times \log_4 5 \times \cdots \times \log_{1023} 1024$$

without using a calculator.

Solution:

Observe that the number above is representable as

$$N = \frac{\ln 3}{\ln 2} \times \frac{\ln 4}{\ln 3} \times \frac{\ln 5}{\ln 4} \times \cdots \times \frac{\ln 1023}{\ln 1022} \times \frac{\ln 1024}{\ln 1023}$$

Simplify the expression to get

$$N = \frac{\ln 1024}{\ln 2} = \log_2 1024 = \log_2 2^{10} = 10$$

Problem 2 *Suppose that ten people enter an elevator at the basement of a building that has* 10 *floors (not counting the basement). Suppose further that there is an equal chance that a person exits the elevator at any of the ten floors, and that where any one person exits the elevator is independent of where any other person exits the elevator. What is the probability that exactly one person exits the elevator at each of the ten floors?*

Solution:

The probability is given by the ratio between the number of outcomes favorable to the event and the total number of outcomes.

Observe that

$$\text{The total number of outcomes} = 10 \times 10 \times 10 \times \cdots \times 10 = 10^{10}.$$

Now the number of favorable outcomes is given by the number of ways that 10 persons can be permuted since it does not matter each person exits at which floor. So

$$\text{The number of favorable outcomes} = 10 \times 9 \times 8 \times \cdots \times 2 \times 1 = 10!$$

The final answer is

$$\frac{10!}{10^{10}} \qquad \text{or} \qquad 0.00036$$

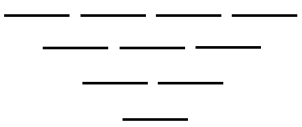
with approximation.

Problem 3 *The base of a triangle is* 80*, and one of the base angles is* 60° *. The sum of the lengths of the other two sides is* 90*. Find the shortest side.*

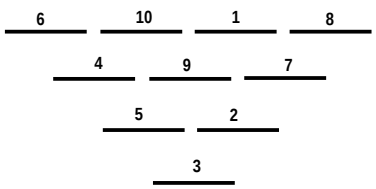
Solution:

Let the triangle be ABC, with $AB = 80$, $BC = a$, $CA = b = 90 - a$. $B = 60^\circ$. Let CD be the altitude to AB. Let $x = BD$. Then $CD = \sqrt{3}x, a = 2x, b = 90 - 2x$. Thus $3x^2 + (80 - x)^2 = (90 - 2x)^2$. So $x = 17/2$; and $a = 17, b = 73$. The shortest side is $BC = 17$.

Problem 4 *Place the numbers* 1 *through* 10 *in the blanks so that any number is the absolute value of the difference of the two numbers directly above it.*



Solution:



Note: the solution is not unique.

Problem 5 *How many ways can the letters in the word*

MATHAWARENESSMONTH

be arranged so there are no adjacent vowels?

Find a mathematical solution of this problem.

Solution:

There are 12 consonants: H, H, M, M, N, N, R, S, S, T, T, W.

There are 6 vowels: A, A, A, E, E, O

First permute the consonants in $\frac{12!}{2!2!2!2!2!}$ ways. Then insert the 6 vowels in the 13 available positions before and after the consonants to guarantee non-adjacency. There are the binomial coefficient $\binom{13}{6}$ ways to do this.

Finally, permute the vowels in $\frac{6!}{2!3!}$ ways.

So the final answer is the product:

$$\frac{12!}{2!2!2!2!2!}\binom{13}{6}\frac{6!}{2!3!}$$

Or

$$\frac{12!13!}{2!2!2!2!2!3!7!} = 1,541,187,648,000$$