NUMERICAL RANGES OF MATRICES

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Introduction

Our purpose is to introduce and illustrate the notion of *Numerical Range* in as few words as possible.

The Definition

Given an $n \times n$ square matrix $A = (a_{ij})_{i,j=1,\ldots,n}$ with complex entries, the numerical range $W(A)$ of A is the subset of all complex numbers of the form

$$w = \sum_{i,j=1}^{n} a_{ij} z_j \overline{z_i} \quad \text{calculated for all } n\text{-tuples}$$

$$z = (z_1, z_2, \ldots, z_n) \quad \text{such that} \quad \sum_{k=1}^{n} |z_k|^2 = 1.$$

In other words, the numerical range of A is the image of the unit sphere of $\mathbb{C}^n$ under the quadratic form associated to A.

A First Example

THE ELLIPTIC RANGE THEOREM:

The numerical range of a 2×2 matrix is necessarily a, (possibly degenerate) closed elliptical disk. The foci of the ellipse, (in the non degenerate case) are the eigenvalues of the matrix.

Figure 1 represents the elliptical numerical range of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2i \end{pmatrix}.$$

The cartesian equation of the ellipse is

$$15 - 36x + 20x^2 - 24y + 16xy + 8y^2 = 0$$

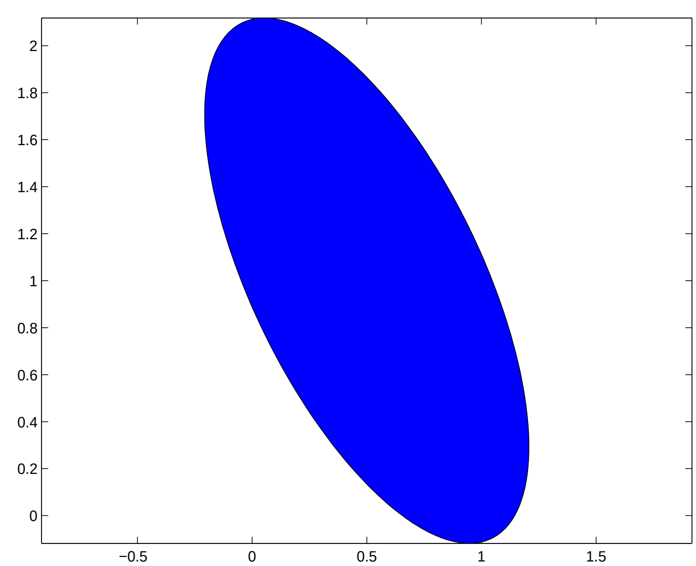

Figure 1.

3×3 **Matrices**

Compared with the previous example, the diversity of shapes numerical ranges of 3×3 matrices can exhibit is surprising. Using a technique of Kippenhahn one can classify their shapes in the following categories.

- Elliptic
- Triangular or Cone-like
- Corner-free with a flatness
- Ovular

Figures 2-8 illustrate this classification.

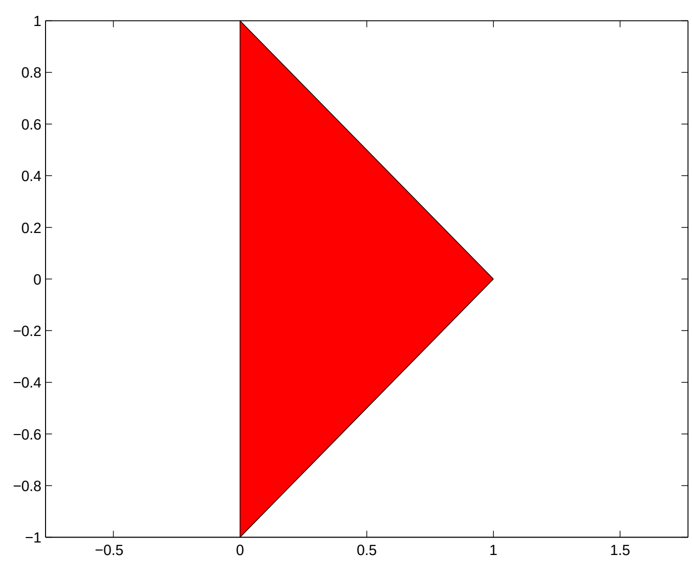

Figure 2. Numerical Range of [1 0 0; 0 i 0; 0 0 -i].

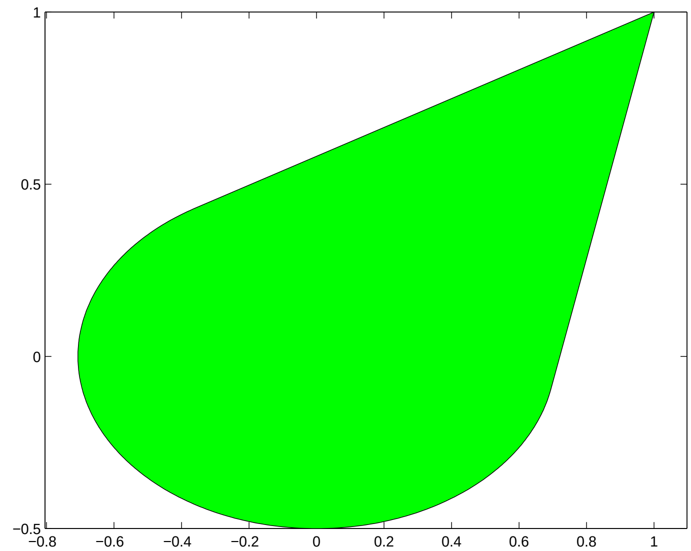

Figure 3. Numerical Range of [1+i 0 0; 0 1/2 i; 0 0 -1/2].

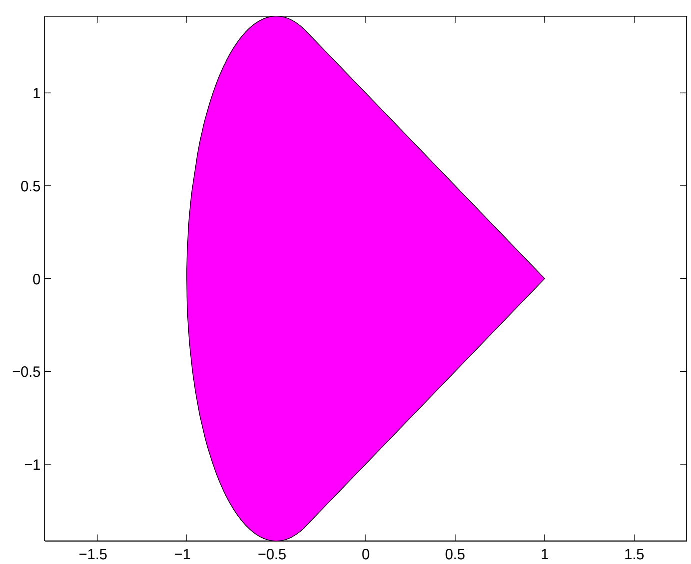

Figure 4. Numerical Range of [1 0 0; 0 i 0; 0 0 -i].

Numerical Ranges with a Flatness

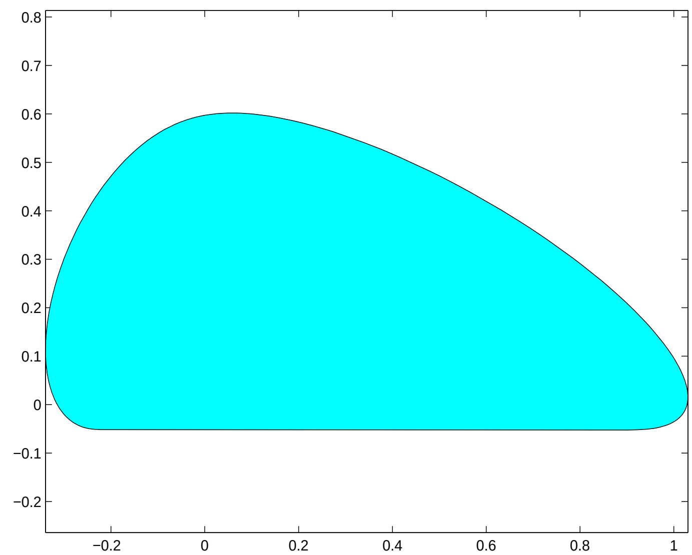

Figure 5. Numerical Range of [1 0 0; i/3 i/2 0; -1/9 -1/3 -1/4].

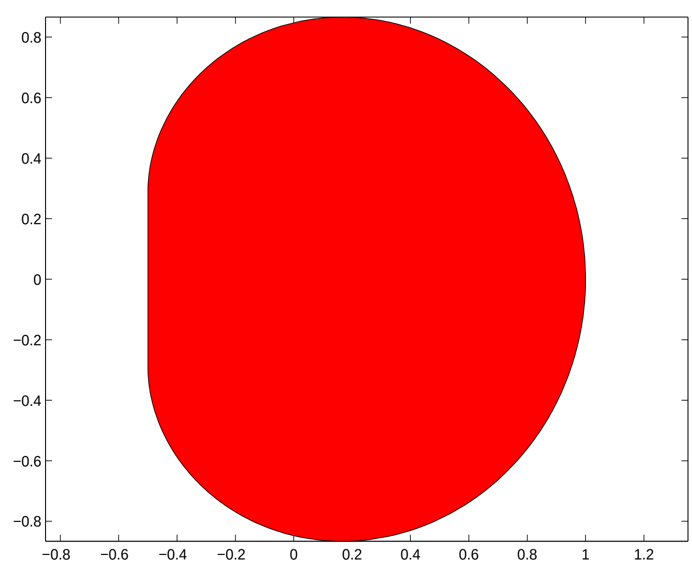

Figure 6. Numerical Range of [0 1 1; 0 0 1; 0 0 0].

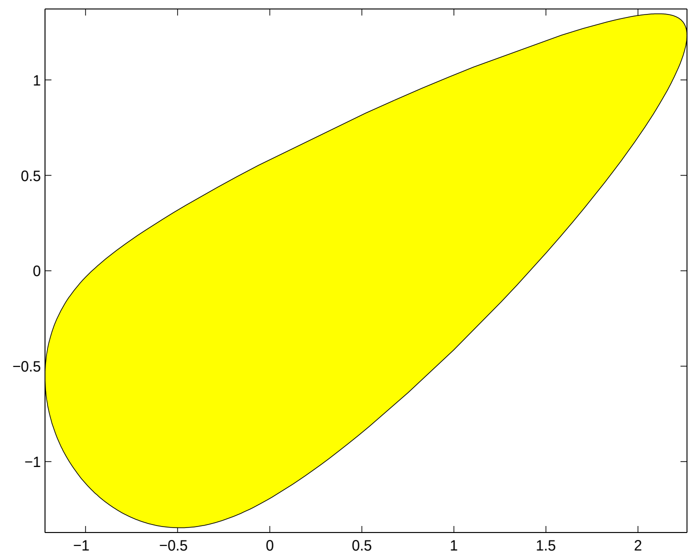

Figure 7. Numerical Range of [1 i i+1; 1/2 0 1; i+2 0 0].

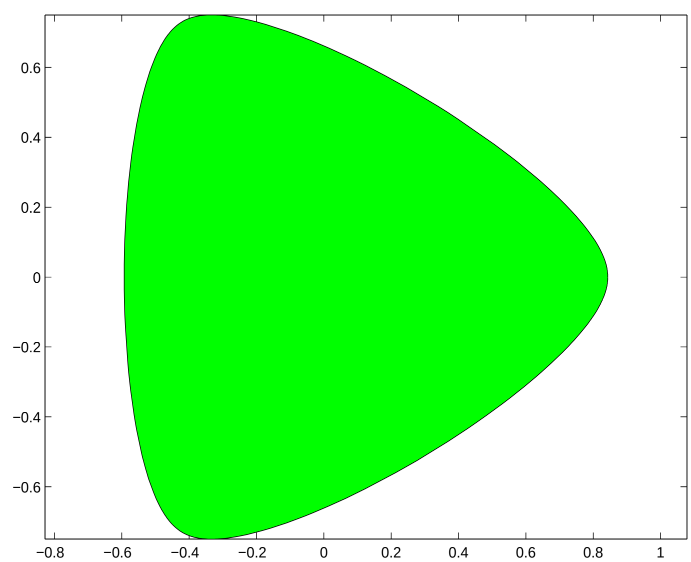

Figure 8. Numerical Range of [1 0 0; 0 i 0; 0 0 -i].

On Kippenhahn's Method

While Kippenhahn's 1951 paper contains the method to construct a curve which can be used to find the equation of the boundary of the numerical range of any matrix, the system of coordinates in which the Kippenhahn Curve looks good is not the cartesian one. In order to convert to cartesian coordinates we used a formula found by Fiedler in 1981. There are some limitations of both Kippenhahn's and Fiedler's formulas. For instance a simple matrix like the one in Figure 8 has a Kippenhahn Curve whose cartesian equation calculated with Fiedler's formula looks as follows.

$$1701/65536x^4y^2 + 729/131072x^2 + 729/32768x^5 + 729/131072y^2 - 729/65536x^6 - 27/65536 - 8991/1048576y^4 + 1539/65536y^2x - 4779/65536x^2y^4 - 297/65536y^6 - 8991/524288x^2y^2 - 2187/32768xy^4 - 513/65536x^3 - 729/16384x^3y^2 - 8991/1048576x^4 = 0$$

Scary, right? To say nothing that if one uses MAPLE to graph the curve whose cartesian equation is displayed above one will have the surprise of producing the output in Figure 9 which shows that the Kippenhahn equation can produce nested curves, only one of which is the boundary of the numerical range. Thus the cartesian system of coordinates seems a little unsuitable for the problem of writing the equation of the boundary of the numerical range.

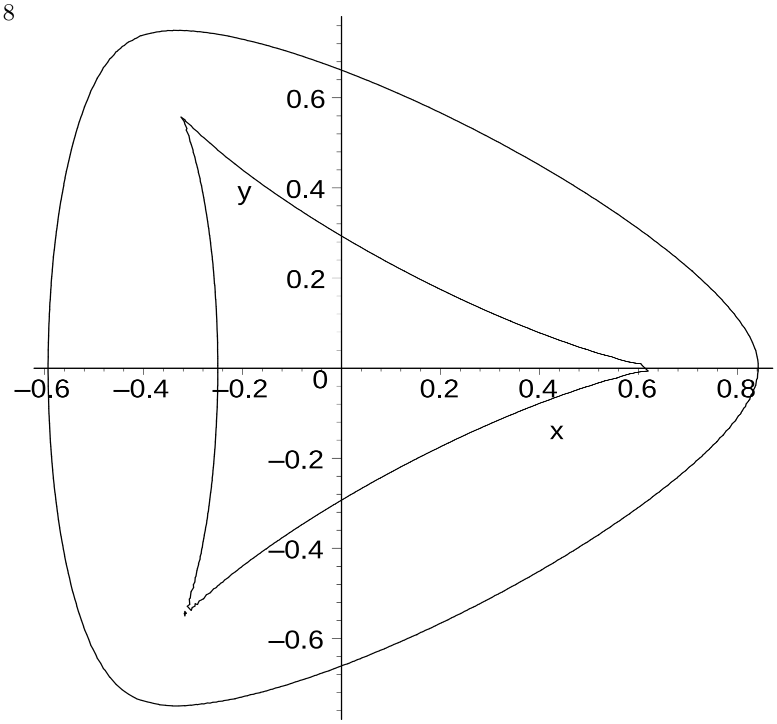

Figure 9: The Kippenhahn curve for the matrix in Figure 8.

Larger matrices can exhibit a larger variety of shapes. To give a nice example Figure 10 illustrates the numerical range of the following 6×6 matrix.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

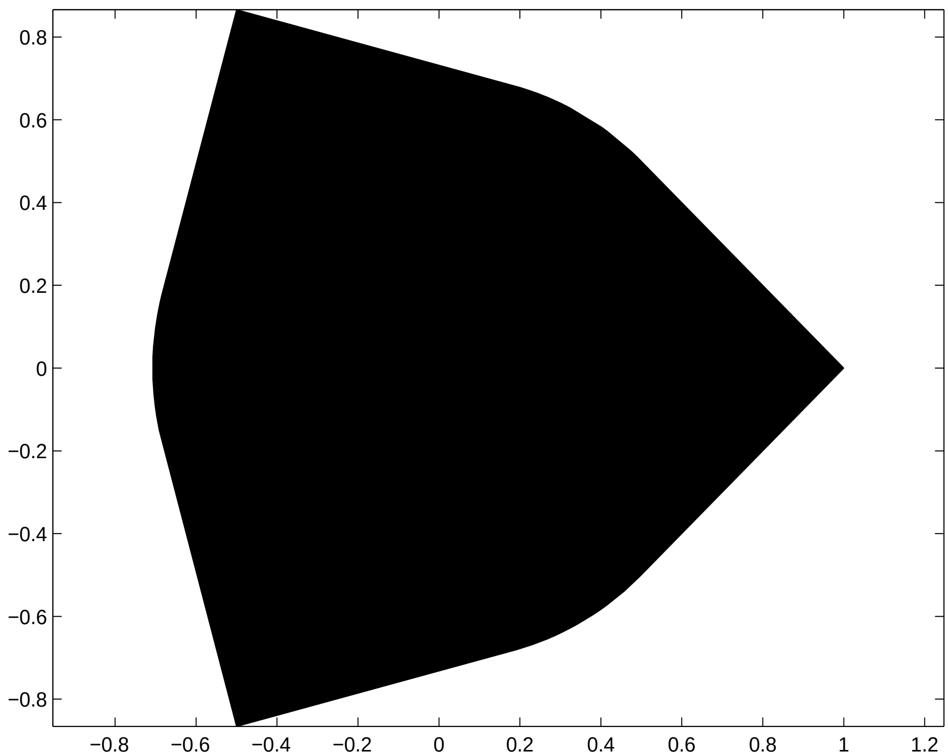

Figure 10: Numerical Range of a 6×6 Matrix

What do all numerical ranges have in common? Although diverse they are all bounded subsets of the complex plane, which is rather easy to establish. The other feature is contained by the so called

TOEPLITZ-HAUSDORFF THEOREM:

Numerical ranges are convex subsets of the complex plane.

The above theorem was proved by O. Toeplitz and F. Hausdorff in 1918-1919. It is probably the most famous result on numerical ranges.

The Direct Sum Principle

When a square matrix A consists of square diagonal matrix blocks

$$A = \begin{pmatrix} D_1 & 0 & 0 & \dots & 0 \\ 0 & D_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & D_n \end{pmatrix}$$

one can prove that the numerical range of A is the convex hull of the union of the numerical ranges of the diagonal blocks, (i.e. is the smallest convex set containing the numerical ranges of all diagonal blocks). Whenever this happens one says that A is decomposable in an orthogonal direct sum and writes

$$A = D_1 \oplus D_2 \oplus \cdots \oplus D_n.$$

To illustrate this principle consider the problem of finding the numerical range of

$$A = \begin{pmatrix} 1 & i & i+1 & 0 & 0 & 0 \\ 1/2 & 0 & 1 & 0 & 0 & 0 \\ i+2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Observing that A is the direct sum of the matrices in figures 6 and 7 the solution is Figure 11.

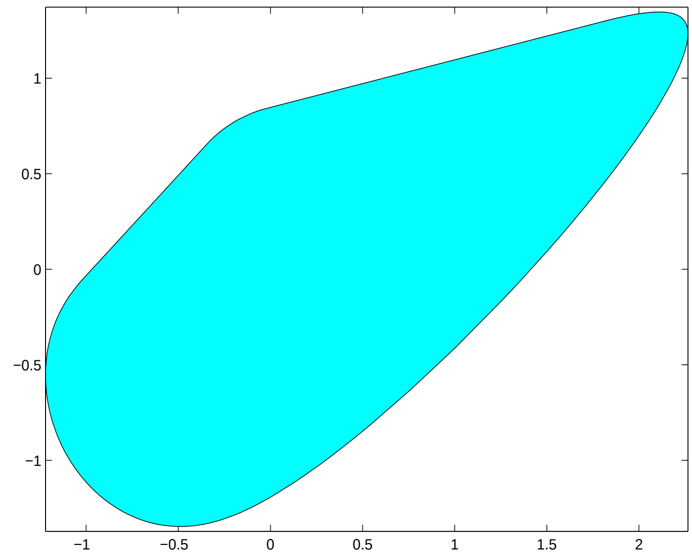

Figure 11. Numerical Range of A.

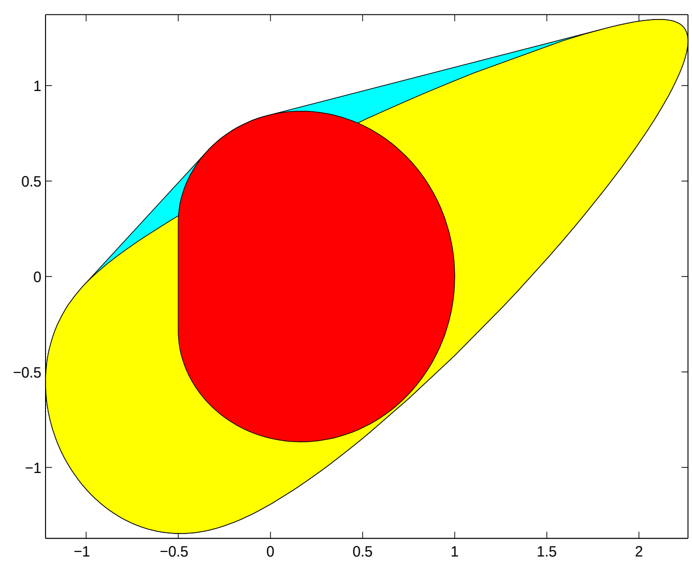

Figure 12. Nested Numerical Ranges of A and its direct Summands.

The Kronecker Product

Let $A = (a_{ij})_{i,j=1,\ldots,n}$ and B be an arbitrary $k \times k$ matrix with complex entries. The Kronecker product $A \otimes B$ of A and B is by definition the $nk \times nk$ block matrix described below

$$A \otimes B = A = (a_{ij}B)_{i,j=1,\ldots,n}.$$

While the Kronecker product is a non commutative operation, it can still be proven that $A \otimes B$ and $B \otimes A$ have identical numerical ranges, and when one of the matrices A and B, say B, happens to be *normal*, (i.e. it commutes with its transpose conjugate) then the following useful formula holds

$$W(A \otimes B) = \mathrm{co}(\sigma(B)W(A)) = \mathrm{co}(W(A)W(B)).$$

Above $\sigma(B)$ denotes the set of all eigenvalues of B, the product set $\sigma(B)W(A)$ consists of all complex numbers representable as products of a number in $\sigma(B)$ and one in $W(A)$, and the particle "co" means that the convex hull of the product set should be calculated. To give an example of how one can use this formula, consider the problem of finding the numerical range of the matrix $C = A \otimes B$ with

$$A = \begin{bmatrix} -\frac{1}{3} & -1 & -1 \\ 0 & -\frac{1}{3} & -1 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} \frac{1+i}{2} & 0 & \frac{-1+i}{2} & 0 \\ 0 & -1 & 0 & 0 \\ \frac{-1+i}{2} & 0 & \frac{1+i}{2} & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}.$$

The 12×12 matrix C looks as follows.

$$C = \begin{bmatrix} \frac{-1-i}{6} & 0 & \frac{1-i}{6} & 0 & \frac{-1-i}{2} & 0 & \frac{1-i}{2} & 0 & \frac{-1-i}{2} & 0 & \frac{1-i}{2} & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1-i}{6} & 0 & \frac{-1-i}{6} & 0 & \frac{1-i}{2} & 0 & \frac{-1-i}{2} & 0 & \frac{1-i}{2} & 0 & \frac{-1-i}{2} & 0 \\ 0 & 0 & 0 & \frac{i}{3} & \frac{-1-i}{6} & 0 & \frac{1-i}{6} & 0 & \frac{-1-i}{2} & 0 & \frac{1-i}{2} & 0 \\ 0 & 0 & 0 & \frac{i}{3} & 0 & 0 & 0 & i & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & \frac{-1-i}{6} & 0 & \frac{1-i}{6} & 0 & \frac{-1-i}{2} & 0 & \frac{1-i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-i}{6} & 0 & \frac{-1-i}{6} & 0 & \frac{1-i}{2} & 0 & \frac{-1-i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{3} & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1-i}{6} & 0 & \frac{1-i}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-i}{6} & 0 & \frac{-1-i}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{3} \end{bmatrix}.$$

One can check that matrix B is normal with set of eigenvalues $\{-1, -i, 1, i\}$. Also if one denotes by I the 3×3 identity matrix and by M the matrix in Figure 6, then $A = (-1/3)I - M$ which means that $W(A)$ is the numerical range of M rotated by an angle of $180°$, (i.e. multiplied by -1), then shifted by $1/3$ to the left, so $W(A)$ has the same shape as $W(M)$ but it lies in a different position. Multiplication be ± 1 and $\pm i$ results in its rotation around the origin by angles of $0°$, $90°$, $180°$, and $270°$ respectively. Therefore the product set $\sigma(B)W(A)$ looks like in Figure 13 and hence its convex hull, $W(A \otimes B) = W(B \otimes A)$ is illustrated in Figure 14.

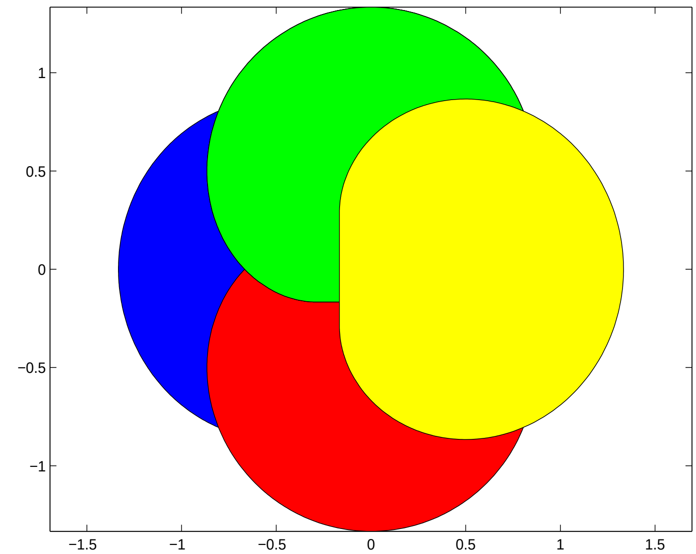

Figure 13: $\sigma(B)W(A)$

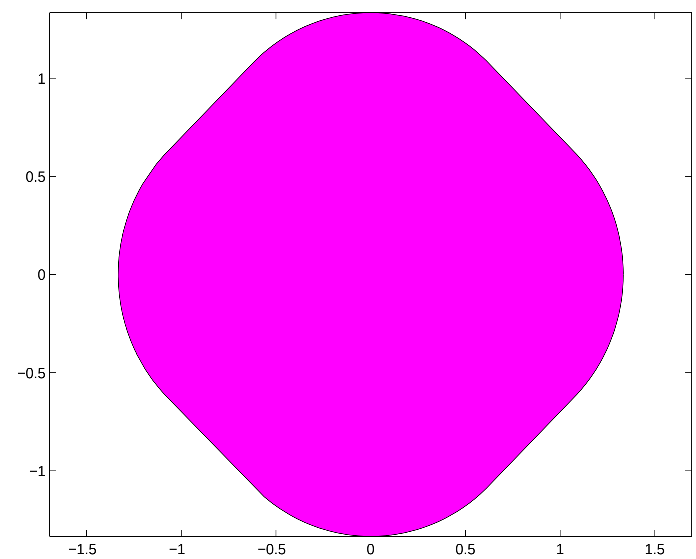

Figure 14: $W(C) = W(A \otimes B) = W(B \otimes A)$

Infinite Matrices

One can associate to each continuous linear operators acting on an infinite dimensional, complex, Hilbert space a matrix. This matrix is not finite though. The numerical range of such an operator is by definition the numerical range of its matrix. The sums involved in the definition of the numerical range need to be convergent infinite series in such a case. Let us consider an example. On the space of all analytic functions $f(z)$ on the open unit disk, with power series expansions

$$\sum_{n=0}^{\infty} c_n z^n \quad \text{such that} \quad \sum_{n=0}^{\infty} |c_n|^2 < \infty.$$

consider the composition operator C_ϕ of symbol $\phi(z) = (z+1)/2$, that is consider the transform C_ϕ mapping each function f in this space to $f \circ \phi(z) = f((z+1)/2)$. A standard basis of this space is the one consisting of all monomials $\{1, z, z^2, z^3, \ldots, z^n, \ldots\}$. Thus for arbitrary n the n-th column in the matrix of C_ϕ consists of the coefficients of the polynomial $z^n \circ \phi(z) = (1+z)^n/2^n$ followed by infinitely many zeros. In other words the n-th column is the n-th row in the Pascal triangle divided by 2^n. Here is the matrix of C_ϕ.

$$C_\phi = \begin{pmatrix} 1 & 1/2 & 1/2^2 & 1/2^3 & 1/2^4 & 1/2^5 & \cdots \\ 0 & 1/2 & 2/2^2 & 3/2^3 & 4/2^4 & 5/2^5 & \cdots \\ 0 & 0 & 1/2^2 & 3/2^3 & 6/2^4 & 10/2^5 & \cdots \\ 0 & 0 & 0 & 1/2^3 & 4/2^4 & 10/2^5 & \cdots \\ 0 & 0 & 0 & 0 & 1/2^4 & 5/2^5 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1/2^5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

The Toeplitz-Hausdorff theorem is valid for this kind of matrices as well, so the numerical range of the matrix above will necessarily be a convex subset of the complex plane. The question is what would be its shape? Here is how one can get some information. By truncating the infinite matrix above to its upper left corner one can obtain a sequence of 2×2, 3×3, 4×4, $\ldots$ matrices namely

$$\begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix} \quad \begin{pmatrix} 1 & 1/2 & 1/2^2 \\ 0 & 1/2 & 2/2^2 \\ 0 & 0 & 1/2^2 \end{pmatrix} \quad \begin{pmatrix} 1 & 1/2 & 1/2^2 & 1/2^3 \\ 0 & 1/2 & 2/2^2 & 3/2^3 \\ 0 & 0 & 1/2^2 & 3/2^3 \\ 0 & 0 & 0 & 1/2^3 \end{pmatrix} \cdots$$

A computer can graph the numerical ranges of these matrices on the same screen, illustrating something rather easy to accept and not so hard to prove, namely that they converge in a certain sense to the infinite matrix of C_ϕ, for which reason their individual numerical ranges will tend to intimately approximate $W(C_\phi)$ as $n \to \infty$. This phenomenon is illustrated in Figure 15 where the boundaries of the numerical ranges of 200 such truncations are graphed.

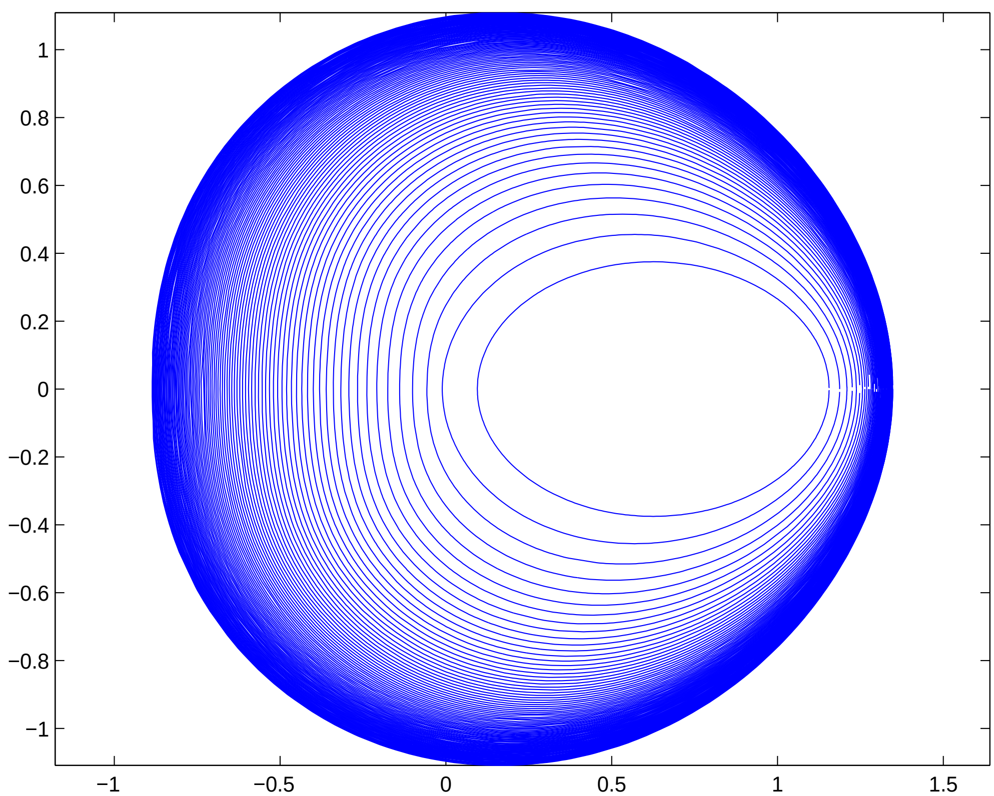

Figure 15: An approximation of $W(C_\phi)$

The problem of finding the exact description of the numerical range of a matrix is not easy, not even in the case of small matrices. Therefore it should not be surprising that, although one can get an idea about the shape of the numerical range of the composition operator above, its exact mathematical description is currently not known. The fact is, that very few composition operators have known numerical ranges. Our interest in numerical ranges originates in our research on numerical ranges of composition operators.

Final Comments

If, as we hope, we got the reader of this mathematical poster interested in the fascinating subject of numerical ranges, here are some bibliographic references.

For an introduction to the subject, take a look at:

K. E. Gustafson and D. K. M. Rao, *Numerical Range*, Springer-Verlag, New York, Heidelberg, Berlin, 1997.

and

P.R. Halmos , *A Hilbert Space Problem Book*, 2-d edition, Springer-Verlag, New York, Heidelberg, Berlin, 1980.

For the little that is known on numerical ranges of composition operators, see the only 2 published papers on this subject:

P.S. Bourdon and J.H. Shapiro, *The Numerical Ranges of Automorphic Composition Operators*, J. Math. Analysis and Appl., 251(2000), 839-854.

and

V. Matache, *Numerical Ranges of Composition Operators*, Linear Alg. and Appl., 331(2001), 61-74.

Finally, if you liked the pictures illustrating the material exposed in this poster, get a copy of MATLAB then go to the MATLAB file-exchange site

http://www.mathworks.com/matlabcentral/fileexchange/index.jsp

click on Mathematics, then on Linear Algebra, and download the routine called **nrange** which we wrote and published there electronically. All the pictures, except Figure 9 were produced with it.