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Fractals and the Cantor Set

1. introduction

Our goal is to investigate the properties of the Cantor Set . We will give a brief description of fractals, define the Cantor Set, show that it is a fractal with zero length, and that it is closed, bounded, perfect and homeomorphic to the powerset of $\mathbb{N}$.

2. What is a Fractal?

Definition 1. *A Fractal is a geometric object that possesses the following two properties:*

(1) *It is self-similar*[1]
(2) *It can possess non-integer dimensions*

So a fractal is an object or quantity which displays self-similarity, in a somewhat technical sense, on all scales. The object need not exhibit exactly the same structure at all scales, but the same "type" of structures must appear on all scales. [2]

3. The Cantor set

Georg Cantor, a mathematician from the late 19th to early 20th centuries, introduced us to a contrived fractal which has come to be known as the Cantor

[1] See Figure 2

[2] A plot of the quantity on a log-log graph versus scale then gives a straight line, whose slope is said to be the fractal dimension. The prototypical example for a fractal is the length of a coastline measured with different length rulers. The shorter the ruler, the longer the length measured, a paradox known as the coastline paradox.see http://mathworld.wolfram.com/Fractal.html for more info on basic fractals

Set, or, for reasons which will become obvious, Cantor Dust. The set has some interesting properties which have led to further research and discovery in fractals and chaos theory. The so called Cantor set is a famous construction in mathematics, much older than the relatively recent interest in Chaos and fractal geometry. It too is simple to describe (although it is not quite so easy to investigate!), and it is perhaps the simplest example of a fractal.

3.1. How is the Set Constructed? [3]

The general approach for constructing the set is by starting with a line segment, and then removing the middle third of the segment. This leaves two subsegments. The geometric construction is iterated by removing the middle thirds of these, and so on.

Formally, the Cantor set has the following definition:

Definition 2. *To define the Cantor set in R, we first define the following sequence of subset of R* $C_0, C_1, \ldots,$ *that satisfy the following conditions:*

(1) $C_0 = [0, 1]$
(2) *Each* C_n *is the union of* 2^n *closed intervals and*

$$C_0 \supset C_1 \supset \cdots \supset C_n \supset C_{n+1} \ldots$$

[3] See Figure 1 for maple generated graph of the Cantor Set

(3) *C_n is constructed by removing the open middle third of each interval in C_{n-1}, i.e., replacing each $[a, b]$ in C_{n-1} by two closed intervals $L[a, b] = [a, a + \frac{1}{3}(b - a)]$ $R[a, b] = [a + \frac{2}{3}(b - a), b]$*

(4) *Then we let $\mathcal{C} = \bigcap_n^\infty C_n$ be the Cantor set.*

The object that remains when this geometric construction has been iterated into the transfinite is the Cantor set.

4. The Topology of the Cantor Set

Let's take a deeper look at the structure of the Cantor set. If we are to get at the bones of the Cantor set, we'll need some definitions, so get out your notebooks because there may be a quiz.

Definition 3. *A point p of a subset P of $\mathbb{R}^n$ is an **isolated point** of P if there is an open set U in $\mathbb{R}^n$ o such that $U \cap P = \{p\}$.*

Definition 4. *A subset S of $\mathbb{R}^n$ is **discrete** if every point is an isolated point in S. We may also characterize a perfect set to be a closed set such that every point is an accumulation point.*

Definition 5. *A nonempty closed subset F of $\mathbb{R}^n$ is said to be **perfect** if it has no isolated points, or rather is identical to its derived set.*

Definition 6. *A collection A subsets of X is said to **cover** X if the union of the elements of A is* *equal to X. It is called an open covering if its elements are open subsets of X.*

Definition 7. *A space is* ***compact*** *if every open covering A of X contains a finite. subcover*

Now that you're an expert in topology and analysis, you will have no problem deriving the theorems below.

Theorem 8. *The Cantor set is compact.*

Proof. Recall the definition of the Cantor set is as follows: we let

$$C_0 = [0, 1]$$

and define, for each n, the set C_n inductively as

$$C_n = C_{n-1} \setminus \bigcup_{n=0}^{\infty} (\frac{1+3k}{3^n}, \frac{3+3k}{3^n})$$

then the Cantor set is given by

$$C = \bigcap C_n$$

Observe that each set

$$\bigcup_{n=0}^{\infty} (\frac{1+3k}{3^n}, \frac{3+3k}{3^n})$$

is open. Clearly [0,1] is a closed set since it is a closed interval. Recall, that by De Morgan's we know that the complement of an open set is a closed set so the set C_n are closed as well. This follow by an easy application of mathematical induction. Furthermore, observe that each set C_n is contained in C_0, i.e., the sequence is monotonic decreasing. This tells us the each C_n is bounded.

Subsequently, we know that C is the intersection of closed, bounded sets, and thus it also closed and bounded. But, this is exactly what it means for a set to be compact, so we have proved the theorem.

Theorem 9. *The Cantor set is perfect.*

The fact that the Cantor set is perfect implies that it is uncountable. This is a consequence of a fundamental theorem from analysis:

Theorem 10. *Any perfect subset of $\mathbb{R}^n$ has the power of the continuum.*

The last topological property of the Cantor set we note is in relation to it measure. We'll give a proof since it is fairly straight forward. Here's one more definition you'll need for the test:

Definition 11. *$We give an intuitive definition$ A set $C \subset \mathbb{R}$ is said to be negligible or of* ***measure zero*** *if it can be covered by countably many intervals whose sum of lengths may be arbitrarily small.*

Theorem 12. *The Cantor set has length zero.*

Proof. We use the same characterization as above for the set C. But to be perspicuous let us map out the construction to make thing a little more digestible:

$$C_0 = [0, 1]$$

$$C_1 = [0, 1] \setminus (1/3, 2/3)$$

$$C_2 = C_1 \setminus [[(1/9), (2/9)] \cup [(7, 9), (8, 9)]$$

$$\vdots$$

That is, at the n^{th} stage we remove the 2^{n-1} interval from each previous set, each having length $\frac{1}{3^n}$. Thus, we will remove a total length of

$$\sum_{n=1}^{\infty} 2^{n-1} \frac{1}{3^n} = \frac{1}{3} \sum_{n=1}^{\infty} (\frac{2}{3})^n = 1$$

from the unit interval [0,1]. Now since we remove a set of total length 1 from the unit interval, the length of the remaining Cantor set must be null which is what we wanted to establish. This completes the proof. □

Let's move on. This stuff is fun, but Chaos is more super fun—Dig.[4]

[4]We're math students, leave the grammar to the to the physics department

5. Fractal Dimension

The best way to approach the matter of non-integer dimensions is to examine examples of objects with integer dimensions that we are familiar with. Consider a line segment, which has one dimension. If we break this line segment up into N equally sized portions, where r is the scaling ratio, then we observe that $Nr = 1$. So, for instance, say a line segment one unit in length is divided into 4 equal pieces, then$r = \frac{1}{4}$, and $4(\frac{1}{4}) = 1$. Similarly, we can expand this to a two-dimensional square. If we use the scaling ratio $r = \frac{1}{3}$ such that we divide each of its sides into 3 equal pieces, then this will result in dividing the original square into 9 subsquares. Note that $9(\frac{1}{3})^2 = 1$, and in general $Nr^2 = 1$. If we extend this notion to a three-dimensional object, one can derive the relation $Nr^3 = 1$. Notice that in each case, when we partition the original object into N equal subunits, the exponent of r is the dimension of the object under scrutiny. In general, continuing along these lines, our relation will be

$$Nr^d = 1.$$

We can find a formula for d by doing some simple algebra, and if we do we get

$$d = \frac{\log(N)}{\log(\frac{1}{r})}$$

There are any number of values of N and r such that when inserted into the formula above for d will generate non-integral solutions for d. One such example we wish to explore is the Cantor "middle thirds" set (or Cantor dust).

6. Cantor Dust

The construction of the Cantor dust is as follows: starting with the unit interval $[0, 1]$, use $r = \frac{1}{3}$ to scale the interval into 3 equal pieces. Now remove the open interval in the middle, that is $(\frac{1}{3}, \frac{2}{3})$ [refer to Figure 1]. We continue repeating this process for each remaining segment, id est, scale each remaining line segment by $r = \frac{1}{3}$, and remove the open middle third of the segment. Notice that since we are removing the middle third, we partition this segment into $N = 2$ equal subunits.

We can calculate the fractal dimension of the Cantor dust by plugging $N = 2$ and $r = \frac{1}{3}$ into our formula for d. We obtain

$$d = \frac{\log(2)}{\log(3)} \approx 0.63092975...$$

Also, if we plug in $N = 2$, $r = \frac{1}{3}$, and $d = \frac{\log(2)}{\log(3)}$ into Nr^d then we will indeed obtain 1 for an answer. Thus, the Cantor dust has as its dimension $d \approx 0.63092975$. So, the Cantor set possesses non-integer dimension, which fulfills one condition for being a fractal. We can examine its self-similarity by referring back to Figure 1. Notice if we "zoom in" or focus on half of the first iteration, it looks similar to the original set $[0, 1]$–that is, a straight line of one dimension. Now if we "zoom in" on the second iteration, focusing on half of one half of the first iteration, again we see a straight line, resembling the original set. If we only focus on one half of the entire second iteration, then we see an object similar to that of the first iteration. We can generalize this process like this: if we focus on half $(\frac{1}{2^1})$ of the n^{th} iteration, we will see an object similar to C_{n-1}. If we focus in on one fourth $(\frac{1}{2^2})$ of the n^{th} iteration, then we will see an object similar to C_{n-2}, and so on.

7. Chaotic Maps

A map is a function whose domain space and range space are equal to each other. For a quick example of a familiar function whose domain space and range space are equal, consider $f(x) = x^3$. The function f has as its domain and range the set of all real numbers. Next, we will define what is known as an orbit. If we let x be a point and f be a map, then the orbit of x under f is the set of points $\{x,\ f(x), f^2(x), f^3(x), \ . \ . \ . \ \}$, where $f^2(x)$ represents $f(f(x))$, $f^3(x)$ represents $f(f(f(x)))$, and so on. The starting point of the orbit, x, is termed the initial value of the orbit.

Definition 13. *Assume we have a set X that serves as the domain and range for a map f .*

*Then $f : X \to X$ is said to be **chaotic** on X if the following three conditions are satisfied:*

(1) *f has sensitive dependence on initial conditions.*
(2) *f is topologically transitive.*
(3) *periodic points are dense in X.*

Definition 14. *A function $f : X \to X$ is said to possess the property of **sensitive dependence** on initial conditions if there exists $\delta > 0$ such that, for any $x \in X$ and any neighborhood N of x, there exists $y \in N$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.*

What this means is for any y selected in any interval about x, eventually, after n iterations of these values under the map f, their respective values are no longer arbitrarily close to each other–that is, they are at least some set distance apart.

Next, we wish to introduce the notion of a topologically transitive function.

Definition 15. *A function $f : X \to X$ is said to be **topologically transitive** if for any open sets $U, V \subset X$, there exists $i > 0$ such that $f^i(U) \cap V \neq \emptyset$.*

The fact that a chaotic function is topologically transitive illustrates the fact that the range of a chaotic map is "filled up." That is, a chaotic function is surjective.

The final part of the definition of a chaotic map is that the periodic points of the map $f : X \to X$ are dense in X. First we must define the notion of a

periodic point in X. The period of a point $x \in X$ is the least positive integer k for which $f^k(x) = x$. In general, a point x is called periodic if for some integer n we have $f^n(x) = x$. An extension of periodic points are eventually periodic points. A point x may not be periodic itself; however, if there exists $m > 0$ such that $f^{k+j}(x) = f^j(x)$ for all $j \geq m$, then we say that the point x is eventually periodic of period k.

Now, if we examine the orbit $\{x, f(x), f^2(x), \ldots\}$, it will necessarily have periodic points which are dense in X (so that the range X is filled up). This means that the closure of the set of all periodic points is equal to X, that is to say, these periodic points become arbitrarily close to each other in X.

I see that you are getting bogged down with formalities, so here's some more. Consider the following example of a chaotic map:

$$f(x) = \begin{cases} 3x, & \text{if } -\infty < x \leq \frac{1}{2} \\ 3(1-x), & \text{if } \frac{1}{2} < x < \infty \end{cases}$$

The above chaotic map is defined for all real numbers. It can be quickly verified that the majority of points go to $-\infty$ upon iteration under f. However,

$f : \Lambda \to \Lambda$ describes the "Cantor middle-thirds" set, where Λ is the Cantor set or Cantor dust as described above. One can check through computation that for any $\lambda \in \Lambda$ it is indeed sensitive dependent on initial conditions, f is topologically transitive, and that the

periodic points are dense in Λ, hence f is chaotic on Λ. Other examples of chaotic maps include logistic equations of the form $f(x) = \alpha x(1-x)$. Maps of this kind are known to be chaotic on Λ if $\alpha > 2+\sqrt{5}$. Ok enough of this applied mathematics. What does math have to do with the real world anyway?

8. Is the Cantor Middle Thirds related to any known set?

This is a typical question topologists ask when they encounter new sets. If they can show that two sets are related, they can use what they know about one to make generalizations about the other. This is where homeomorphisms enter the picture. Two sets that are homeomorphic are virtually indistinguishable. So what exactly is a homeomorphism?

Definition 16. *If (X, d_1) and (Y, d_2) are any two metric spaces, we say a function f: $X \longrightarrow Y$ is a homeomorphism if*

1. *f is bijective*
2. *f is continuous on X*
3. *f^{-1} is continuous on Y*

So our goal is to find a homeomorphism between the Cantor Middle Thirds and some other set. Mofazzal Azam of the Theoretical Physics Division in Bombay India came up with such a function, called the itinerary of a point x, S(x), under the Cantor Map $F(x)$, defined as:

$$S(x) = (n_1, n_2, ..., n_k, ...) \ where \ n_j \in S \ iff \ \ F^{n_j - 1}(x) \in I_o = [0, \frac{1}{3}].$$

As an example, consider x = $\frac{1}{3}$.

The $zero^{th}$ iteration of the Cantor Map gives us $\mathrm{F}^0\left(\frac{1}{3}\right) = \frac{1}{3}$. For the first, second, and third iteration, we have $\mathrm{F}^1\left(\frac{1}{3}\right) = 1$, $\mathrm{F}^2\left(\frac{1}{3}\right) = 0$, $\mathrm{F}^3\left(\frac{1}{3}\right) = 0$, and we see for every iteration hereafter, $\mathrm{F}^n\left(\frac{1}{3}\right) = 0$. Therefore, $S\left(\frac{1}{3}\right) = (1, 3, 4, 5, 6, ...)$.

The standard Euclidean metric is associated with the Cantor Middle Thirds and with the power set of the natural numbers we associate the following metric:

$$||\alpha|| = \sum_{n_k \in \alpha} \frac{1}{2^{n_k}} \ where \ \alpha = (n_1, n_2, ...) \subset \mathbb{N}.$$

Azam proved that $S\left(x\right)$ was bijective, continuous on the Cantor Middle Thirds, and that $S^{-1}\left(x\right)$ was continuous on the power set of the natural numbers, hence a homeomorphism. What does this tell us? Well, since S(x) is a homeomorphism, we have the following equivalences:

(1) For each subset A of the Cantor Middle Thirds,

(2) A is open $\iff$ S(A) is open
(3) A is closed $\iff$ S(A) is closed
(4) A is compact $\iff$ S(A) is compact
(5) A is connected $\iff$ S(A) is connected

We also know:

(6) interior points of S(A) = S(interior points of A)
(7) closure of S(A) = S(closure of A)
(8) limit points of S(A) = S(limit points of A)
(9) boundary points of S(A) = S(boundary points of A)

And finally, any other topological properties that apply to the power set of the natural numbers also apply to the Cantor Middle Thirds. What is this good for? Who cares, it's just cool!

9. Conclusion

The Cantor Dust is compact, i.e., closed and bounded, totally disconnected, and perfect. It is perhaps the oldest mathematically contrived fractal, but hardly the oldest in existence. That title belongs to the fractals found in the clouds, trees, and borders of Great Britain.