How Many Continuous Nowhere Differentiable Functions Are There?

Roger Gaul
Nanhee Kim

Introduction

The concept of a function that is both continuous and nowhere differentiable is not simple. The scientist André Marie Ampère first explored this concept in 1806. Ampère was unsuccessful in his attempt to establish that a continuous function could, in fact, also be nowhere differentiable. More than fifty years later, Karl Theodor Wilhelm Weierstrass made a breakthrough in Ampère's research. Weierstrass was able to construct a continuous nowhere differentiable function. Weierstrass' example was used as a template of sorts for other mathematicians who derived other, simpler, functions to illustrate the concept. Bartel Leendert van der Waerden, who based his on the fact that a uniformly convergent sum of continuous functions is continuous, constructed one of the simplest of these examples.

Definitions

In this section, it will be shown that the set in question has Baire Category II, which makes it bigger (in a topological sense) than the set of continuous functions that are differentiable somewhere.

We want to understand the set of continuous nowhere differentiable functions. For this, we can use Baire's Category Theorem. For understanding Baire's Theorem, we need to define a few terms:

- **Definition:** Let S be a metric space with $A \subset S$. We say that a set A in S is **dense in *S*** if $\overline{A} = S$.

- **Definition:** Let S be a metric space with $A \subset S$. A is **nowhere dense in *S*** if $\overline{A}$ contains no ball of S (that is, the interior of $\overline{A}$ is empty).

- **Definition:** Consider the set $A \subset S$. A is a set of the **First Category in *S*** if and only if A is equal to the union of countably many nowhere dense sets in S.

- **Definition:** The set $A \subset S$ is of the **Second Category in *S*** if it is not of the first category.

- **Definition:** In a metric space (X,d), for each $E \subseteq X$, the **diameter** $\delta(E)$ is defined as:
$$\delta(E) = \sup_{x,y \in E} d(x,y) \in [0,\infty)$$

- **Definition:** A topological space X is said to be **Hausdorff** if, for any two distinct x,y in X, there are disjoint open sets U,V containing x,y respectively.

Baire's Theorem:

If (S,d) is a complete metric space or a locally compact Hausdorff space, then the intersection of every countable collection of dense open subsets of S is dense in S.

- **Definition:** A real or complex vector space on which a norm is defined that is complete is a **Banach Space.**
- **Definition:** Let $C(S)$ be the familiar sup-normed Banach space of all continuous functions on the compact Hausdorff space S. A linear subspace A of $C(S)$ is a **function-algebra** if $fg \in A$ whenever $f \in A$ and $g \in A$.

Stone-Weierstrass Theorem:

If A is a closed sub-algebra of $C(S)$ which:

- Contains the constants,
- Separate points on S, and
- Is self-adjoint (that is $\bar{f} \in A$ whenever $f \in A$),

Then $A = C(S)$.

Let us consider the subset of all ND of all continuous nowhere differentiable functions in the Banach space C(T) of all continuous real functions on an interval T, assuming equal values at the endpoints of T endowed with the supremum norm

$$\|f\|_\infty \equiv \sup_{x\in T} |f(x)|$$

C(T) becomes a Banach space.

The Baire Category Theorem and the Stone-Weierstrass Theorem are the main technical ingredients required to show that ND has category II in C(T)

Conclusion

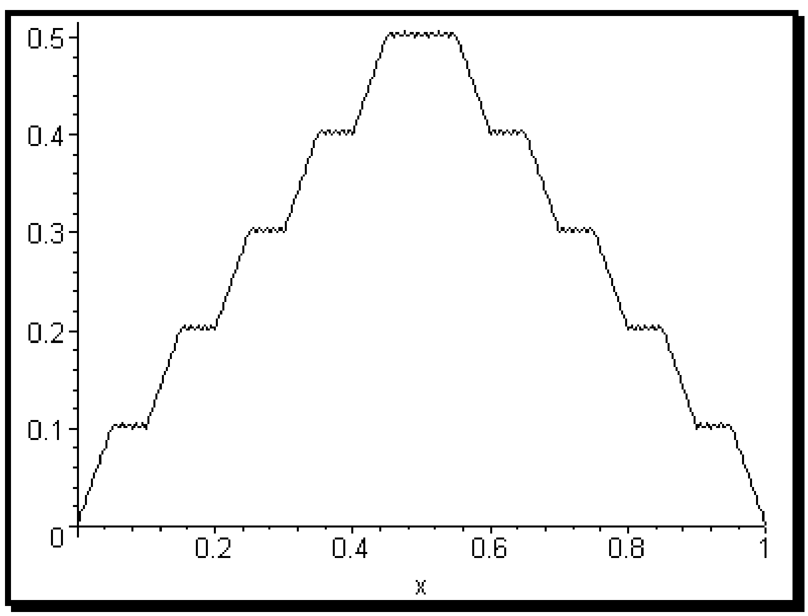
Since the set of nowhere differentiable continuous functions is of Category II, it is much larger than the set of somewhere differentiable continuous functions. There are many more "ugly" functions than there are "nice" ones.

In the following we will illustrate the various examples of continuous, nowhere differentiable functions we know.

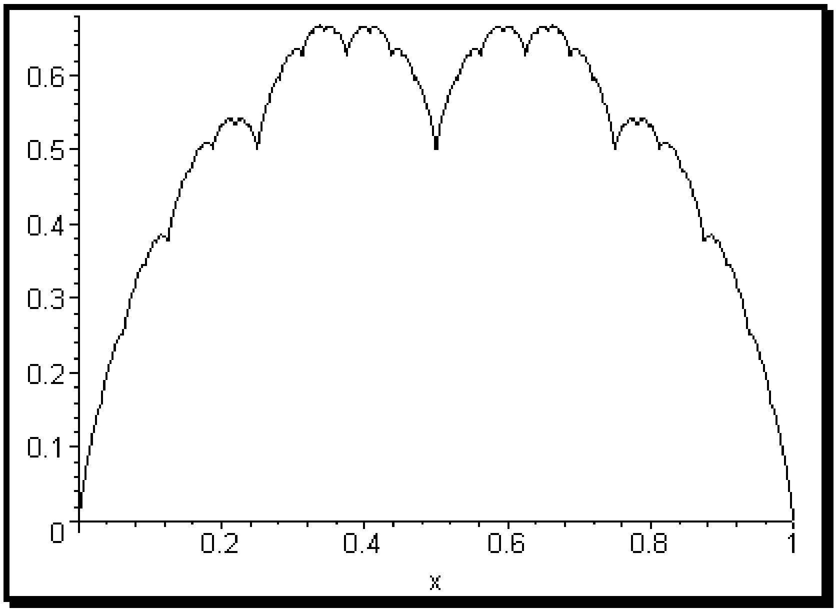
Van der Waerden's Function

$$f(x) = \sum_{n=0}^{\infty} \frac{\{10^n x\}}{10^n}$$

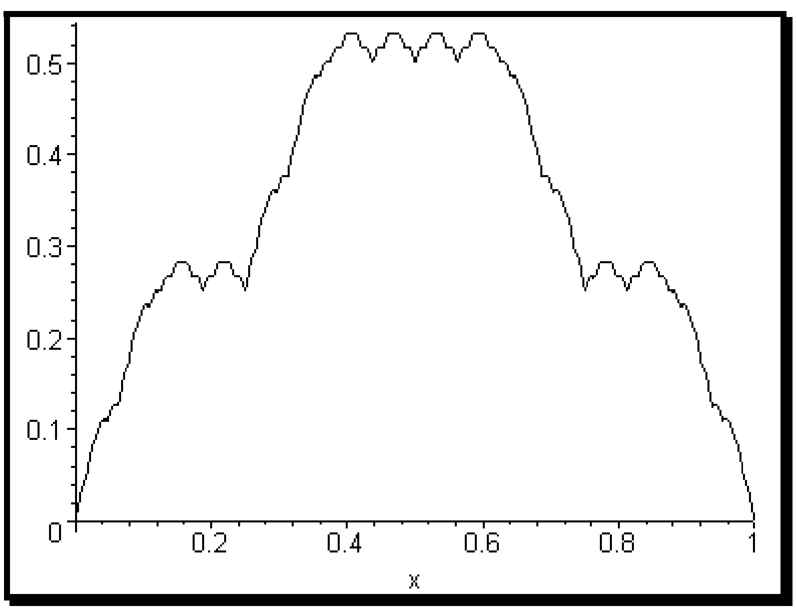
The plot of van der Waerden's function.



A function from Wade's book (see references) is similar in structure to the van der Waerden example.



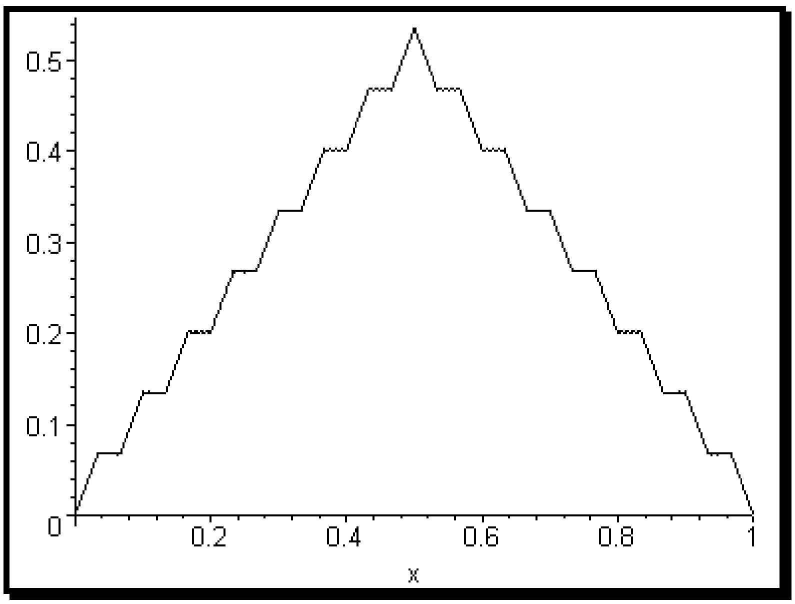
Gelbaum has three example functions. The first is again similar to van der Waerden's example



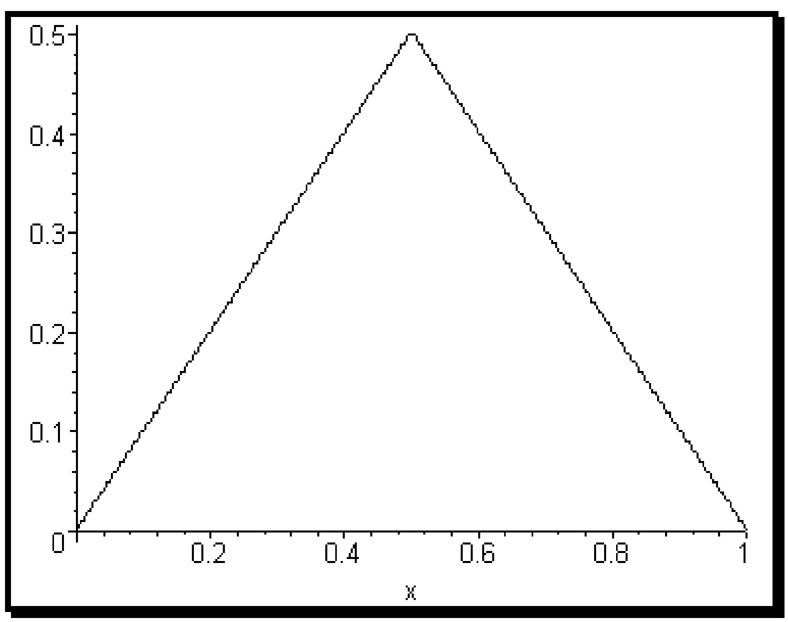
A generalization of van der Waerden's function look like this.

$$f(x) = \sum_{n=0}^{\infty} \frac{\{R^n x\}}{R^n}$$

The examples from Wade and Gelbaum use R values of 2 and 4 respectively. The following plots are R = 15, and 100.



R = 15

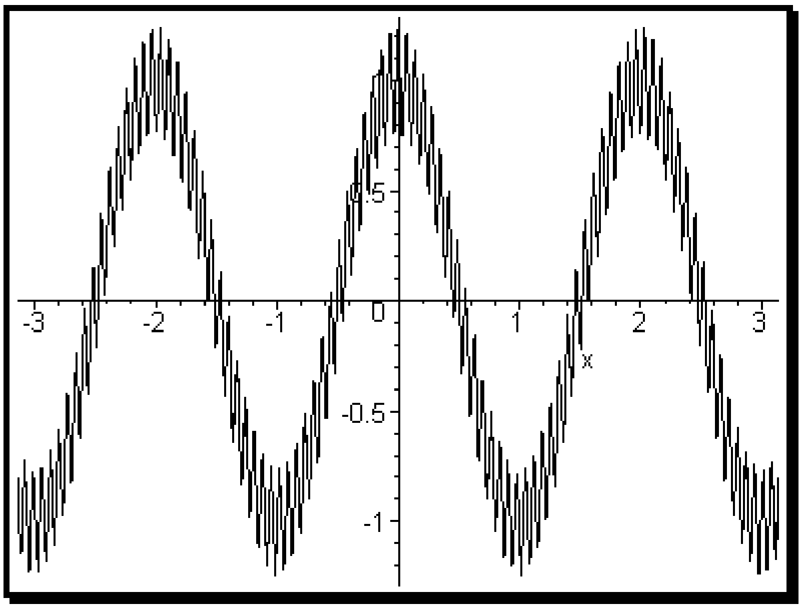


R = 100

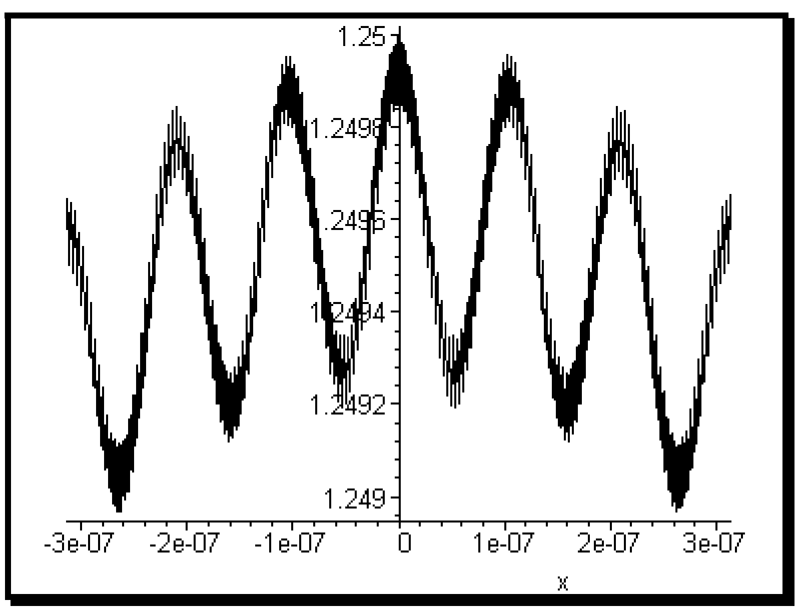
The following function was derived by Weirestrass and illustrates a continuous nowhere differentiable function constructed using a trigonometric function. (Gelbaum pp 50)

$$W(x) = \sum_{k=0}^{\infty} b^k \cos[a^k \pi x]$$

The following is a plot of Weierstass' example with b = .2 and a = 5 + 7.5π. The graph is a cosine wave composed of a repeated wave pattern.



A closer look reveals the underlying structure. Even when the scale is set to $\pm\pi$ 10e-08 similar characteristics are demonstrated:



The following propositions illustrate how nowhere differentiable continuous functions behave when combined in three specific ways. The proofs are elementary and thus left to the reader.

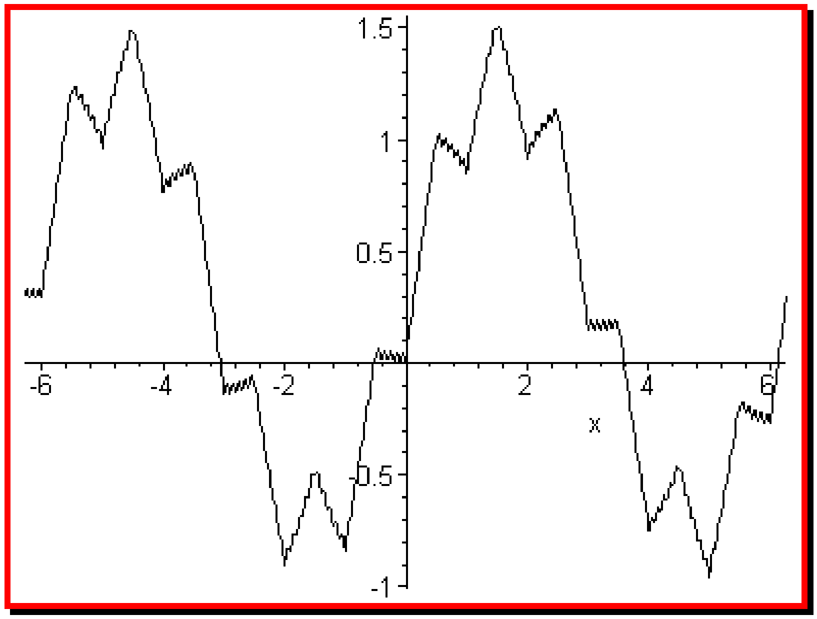
Proposition 1: If $f \in ND$ and F is differentiable on **R**, then $f + F \in ND$.

ND denotes the class of functions continuous and nowhere differentiable on **R**.

Proposition 2: If $f \in ND$, F is differentiable on **R**, and $F(x) \neq 0$, $\forall x \in \mathbf{R}$, then $fF \in ND$.

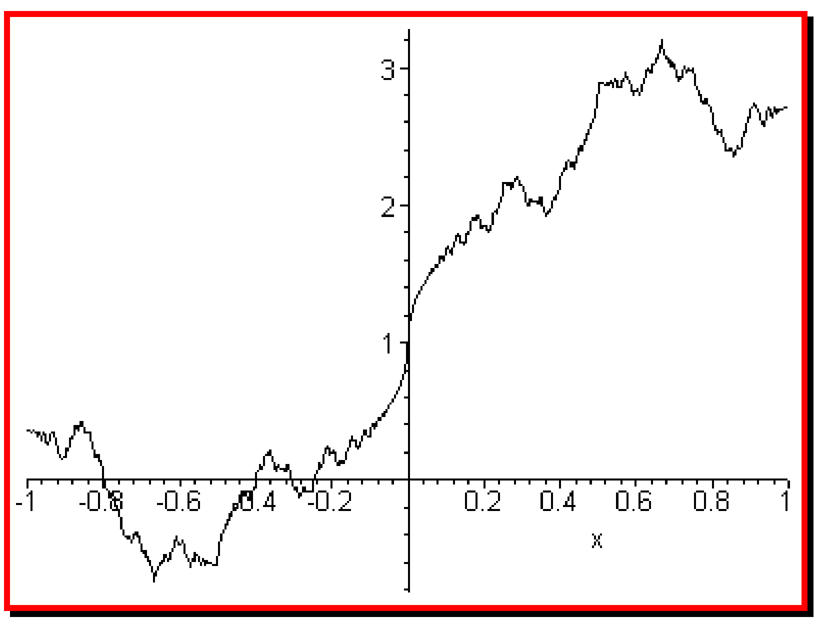
Proposition 3: If $f \in ND$ and F is a continuous bijection with differentiable inverse, then $F \circ f \in ND$.

To illustrate Proposition 1 (the sum of a *ND* function with a differentiable function), the van der Waerden example function is added to the sine function in the plot below.

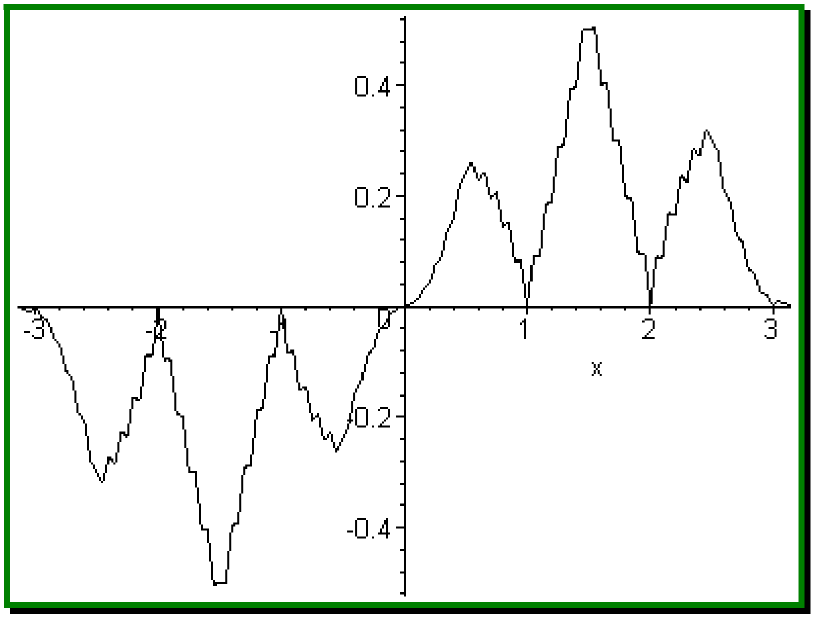


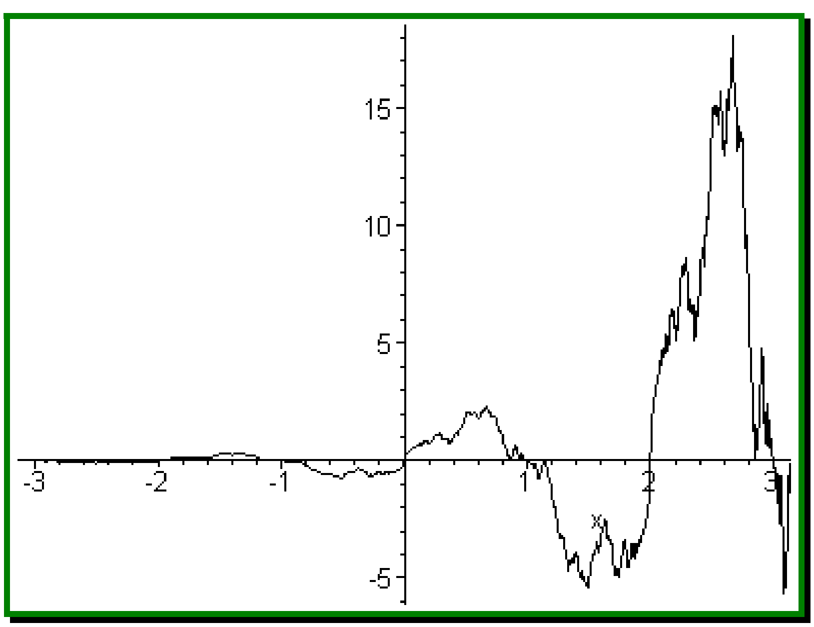
The next plot shows the sum of the nowhere differentiable function below and the exponential function:

$$R(x) = \sum_{n=1}^{\infty} \frac{\sin[xn^2\pi]}{n^2}$$

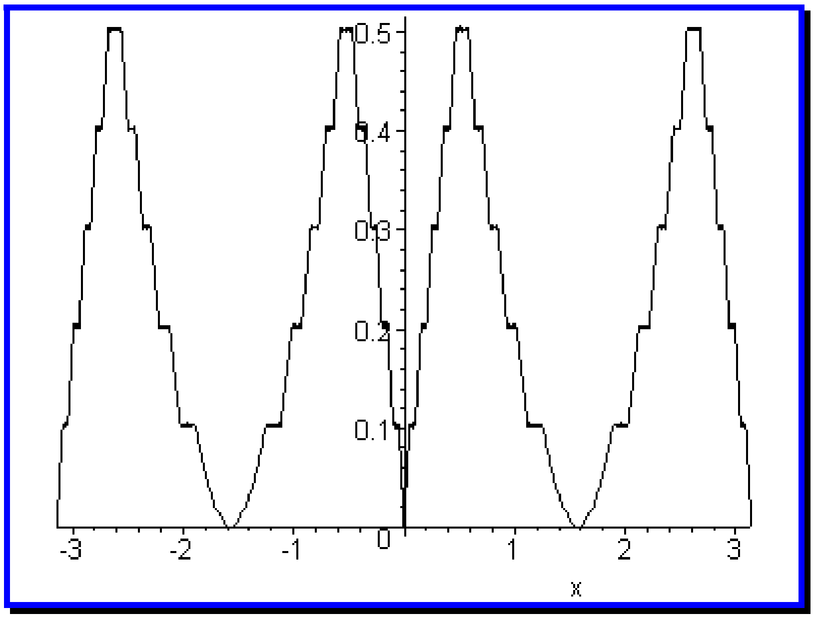


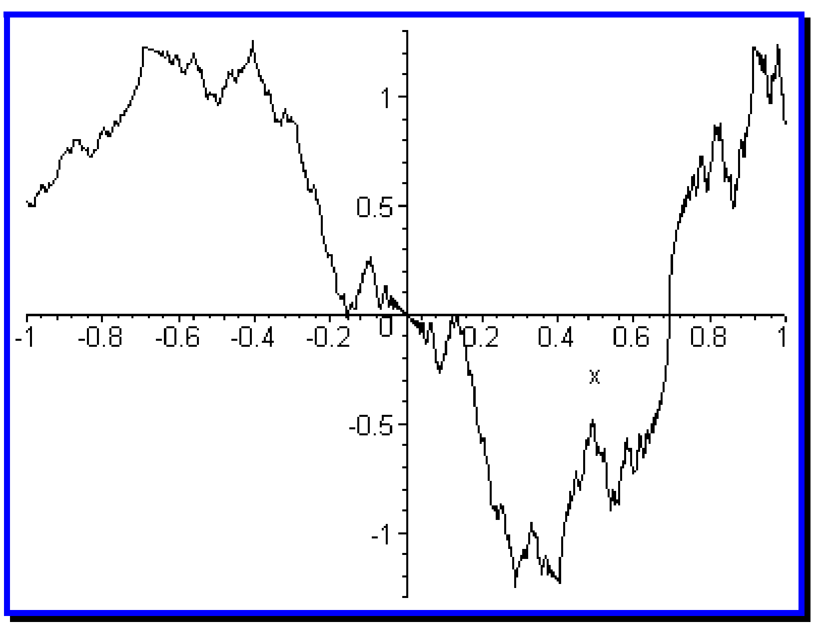
Again the same pairings of functions are used to illustrate Proposition 2, the product of a nowhere differentiable function and a differentiable function is nowhere differentiable.





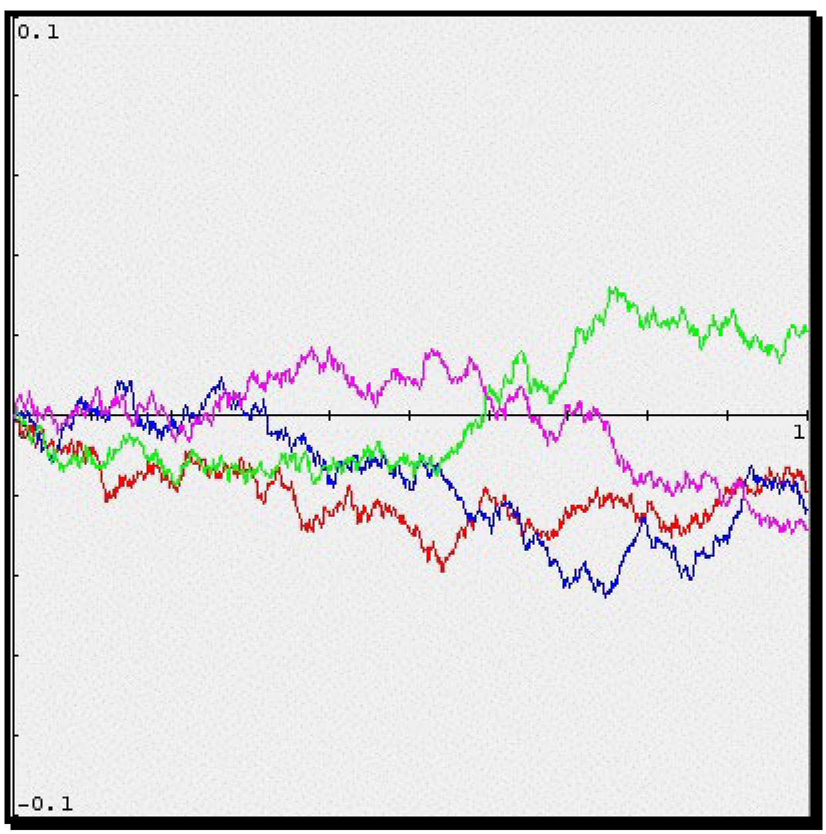
And finally the composition of a nowhere differentiable function and a differentiable function is nowhere differentiable. The same functions used to illustrate are used to produce a graph illustrating Proposition 3.





Some Applications into Brownian Motion

Brownian motion is the observed movement of small particles as they are randomly bombarded by the molecules of the surrounding medium. This was first observed by the biologist Robert Brown and was eventually explained by Albert Einstein. Brownian motion originally refers to the random motion observed under microscope of a pollen immersed in water. Einstein pointed out that this motion is caused by random bombardment of (heat excited) water molecules on the pollen. Modern theory call it a stochastic process. An approximation (discretization) of the 1 dimensional Brownian motion can be described as a "drunken man wandering around the square". More precisely, each of his steps (in both x- and y- directions) are independent normal random variables. In fact this is what the simulation is performing.



References

[1] Ampère, A.M., *Recherches sur quelques points de la théorie des functions derives*, Ecole Polytechnique, 6(1806), fasc.13.

[2] Du Bois Raimond P., *Versuch einer Classification der wilkürlichen Functionen reeler Argumente*, Journal für Math., 79(1875), 21-37.

[3] Gelbaum, B.R. and Olmsted, J.M.H., *Theorems and Counterexamples in Mathematics*, Springer-Verlag, New York, 1990.

[4] Longman, R. and Thompson, L., *Continuous Nowhere Differentiable Functions*, Class-Project, UNO, 1999.

[5] Protter, M.H. and Morrey, C.B., *A First Course in Real Analysis*, Springer-Verlag, New York, 1991.

[6] Riesz, F. and Sz.-Nagy, B., *Functional Analysis*, Ungar, New York, 1955.

[7] Rudin, W., *Functional Analysis*, McGraw-Hill, 1970.

[8] Waerden, B.L. van der, *Ein einfaches Beispiel einer nichdifferenzierbaren stetigen Funktion*, Math. Zeitschr., 32(1930), 474-475.

[9] Wade, W.R., *An Introduction to Analysis*, Prentice Hall, 2000.