Convergent sequences of composition operators

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Abstract

Composition operators $C_{\psi}$ on the Hilbert Hardy space $H^2$ over the unit disk are considered. We investigate when convergence of sequences $\{\psi_n\}$ of symbols, (i.e., of analytic selfmaps of the unit disk) towards a given symbol $\psi$, implies the convergence of the induced composition operators, $C_{\psi_n} \to C_{\psi}$. If the composition operators $C_{\psi_n}$ are Hilbert–Schmidt operators, we prove that convergence in the Hilbert–Schmidt norm, $\|C_{\psi_n} - C_{\psi}\|_{HS} \to 0$ takes place if and only if the following conditions are satisfied: $\|\psi_n - \psi\|_2 \to 0$, $\int 1/(1 - |\psi|^2) < \infty$, and $\int 1/(1 - |\psi|^2) \to \int 1/(1 - |\psi|^2)$. The convergence of the sequence of powers of a composition operator is studied.

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1. Introduction

In this paper, $U := \{z \in \mathbb{C}: |z| < 1\}$ is the open unit disk, and $T$ its boundary, the unit circle. By $m$ we denote the normalized arc-length measure on $T$. We consider the Hilbert Hardy space $H^2$, consisting of all analytic functions $f$ on $U$ for which

$$\|f\|_2 := \sup_{0<r<1} \left( \frac{1}{2\pi} \int_T |f(r\zeta)|^2 \, dm(\zeta) \right)^{1/2} < \infty. \tag{1}$$

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The quantity in (1) is the norm of $H^2$ and has the alternative description
\[ \| f \|_2 = \sqrt{\sum_{n=0}^{\infty} |c_n|^2}, \] (2)
where $\{c_n\}$ is the sequence of the Taylor coefficients of $f$.

Any $H^2$-function $f$ has a radial limit function defined as
\[ f(\zeta) = \lim_{r \to 1^-} f(r\zeta), \quad \zeta \in T. \]

It is well known that the radial limit function is defined $m$-a.e. on $T$. Throughout this paper, it will be denoted by the same symbol $f$ as the function itself. The $H^2$-norm of any $H^2$-function equals the $L^2$-norm of its radial limit function.

The space $H^\infty$ is the space of all bounded analytic functions $f$ on $U$ endowed with the norm
\[ \| f \|_\infty := \sup_{|z| < 1} |f(z)|. \] (3)

Let $S$ denote the subset of $H^\infty$ consisting of the analytic selfmaps of $U$. Each function $\varphi \in S$, induces a bounded composition operator $C_\varphi$, defined as
\[ C_\varphi f := f \circ \varphi, \quad f \in H^2, \]
and referred to as the composition operator of symbol $\varphi$.

In this paper, our basic problem can be formulated as follows. Consider all composition operators on the Hilbert Hardy space $H^2$. For a sequence of symbols $\{\varphi_n\}$ assume that there is some $\varphi \in S$ so that $\{\varphi_n\}$ converges in some sense to $\varphi$. We will investigate under what circumstances one can deduce that $C_{\varphi_n} \to C_\varphi$ in the sense of some usual convergence-concept for sequences of operators.

The problem under investigation is not interesting if the space $L(H^2)$ of all linear bounded operators on $H^2$ is endowed with the uniform operator-topology, unless $\varphi$ satisfies the condition
\[ |\varphi(\zeta)| < 1 \quad m$-a.e. on $T. \] (4)

Indeed, a result by Berkson [1] states that if $E_\varphi = \{\xi \in T: |\varphi(\xi)| = 1\}$ is such that $m(E_\varphi) > 0$, then $C_\varphi$ is an isolated element of the space of all composition operators in $L(H^2)$ endowed with the uniform operator topology. Hence, we are interested mainly in symbols $\varphi \in S$ with the property (4).

Howard Schwartz was the first to consider the problem of relating convergence of symbols and convergence of the induced composition operators. In [6] he settled the cases of weak and strong operator-convergence, proving that $C_{\varphi_n} \to C_\varphi$ in the weak operator topology if and only if $\varphi_n \to \varphi$ weakly in $H^2$, respectively that $C_{\varphi_n} \to C_\varphi$ strongly if and only if $\|\varphi_n - \varphi\|_2 \to 0$. He obtained partial results on uniform convergence too. According to [6, Theorem 4.5], $\|C_{\varphi_n} - C_\varphi\| \to 0$ if $\varphi_n \to \varphi$ a.e.,
\[
\int_{T} \frac{dm}{1 - |\varphi_n|^2} < \infty, \quad n = 1, 2, \ldots,
\]
and
\[ \int_T \frac{dm}{1 - |\psi_n|^2} \to \int_T \frac{dm}{1 - |\psi|^2} < \infty. \]
The operators involved in the statement above are Hilbert–Schmidt composition operators. It is natural to ask if it holds in the stronger sense of Hilbert–Schmidt norm-convergence.

In Section 2 we show that this is the case and, besides improving Schwartz’s result, obtain a necessary and sufficient characterization of the situation when a sequence of Hilbert–Schmidt composition operators \( \{C_{\psi_n}\} \) converges in the Hilbert–Schmidt norm to some composition operator \( C_{\psi} \). More exactly, we prove that this happens if and only if
\[ \int_T \frac{dm}{1 - |\psi|^2} < \infty, \quad \|\psi_n - \psi\|_2 \to 0, \quad \text{and} \quad \int_T \frac{dm}{1 - |\psi_n|^2} \to \int_T \frac{dm}{1 - |\psi|^2}. \]
Recent results of [5] are also extended in Section 2. In Section 3 we consider the sequence \( \{C_{w}\} \) for some non-inner \( \psi \in \mathcal{S} \) having a fixed point \( w \in \mathcal{U} \). We prove that \( \|C_{\psi_k} - C_w\| \to 0 \). Since \( C_w \) is a rank-one idempotent, hence, in particular, a Hilbert–Schmidt operator, it is normal to ask if the previous result can be improved to Hilbert–Schmidt norm convergence, in select cases. We show that this happens if and only if \( C_{\psi_k} \) is Hilbert–Schmidt for some \( k \).

In Section 4 we prove the norm estimate
\[ \|C_{\psi} - C_{\phi}\| \leq 2 \sqrt{\int_T \frac{|\psi(u) - \psi(u)|}{(1 - |\psi(u)|)(1 - |\psi(u)|)} dm(u)}, \]
and deduce that \( \|C_{\psi_n} - C_{\phi}\| \to 0 \) if
\[ \int_T \frac{|\psi_n - \phi|}{(1 - |\psi_n|)(1 - |\psi_n|)} dm \to 0, \]
which relates to a result in Section 2, where it is proved that, if the condition above holds and \( C_{\phi} \) is Hilbert–Schmidt, then \( C_{\psi_n} \) tends to \( C_{\phi} \) in the Hilbert–Schmidt norm.

As a final remark in this introductory section, we would like to observe that, if a sequence of composition operators tends weakly to an operator, then that operator too must be a composition operator. Indeed:

**Remark 1** [6]. The set of all composition operators is weakly sequentially compact.

### 2. Hilbert–Schmidt norm convergence

On any Hilbert space \( \mathcal{H} \), the Hilbert–Schmidt norm \( \|T\|_{\text{HS}} \) of an operator \( T \) is defined as
\[ \|T\|_{\text{HS}} = \sqrt{\sum_{n=0}^{\infty} \|Te_n\|^2}. \]
where \( \{ e_n \} \) is an orthonormal basis of \( H \). The quantity in (5) does not depend on the orthonormal basis chosen \([4]\), thus it is larger than or equal the operator norm \( \| T \| \) of \( T \). Therefore, if we can prove that, under certain assumptions on \( \{ \psi_n \} \), one has that \( \| C_{\psi_n} - C_{\psi} \|_{\text{HS}} \to 0 \), then we can deduce \( \| C_{\psi_n} - C_{\psi} \| \to 0 \).

Recall that a Hilbert-space operator is called a Hilbert–Schmidt operator if it has finite Hilbert–Schmidt norm. It is well known \([7]\), that a composition operator \( C_\psi \) on \( H^2 \) has Hilbert–Schmidt norm given by

\[
\| C_\psi \|_{\text{HS}} = \left[ \int_T \frac{dm}{1 - |\psi|^2} \right]^{1/2}.
\]

(6)

**Lemma 1.** The sequence \( \{ C_{\psi_n} \} \) of Hilbert–Schmidt composition operators tends toward the composition operator \( C_\psi \) in the Hilbert–Schmidt norm if \( \psi_n \to \psi \), a.e. on \( T \),

\[
\int_T \frac{dm}{1 - |\psi_n|^2} < \infty,
\]

and

\[
\int_T \frac{dm}{1 - |\psi_n|^2} \to \int_T \frac{dm}{1 - |\psi|^2}.
\]

(7)

(8)

**Proof.** Using formula (5) and the standard orthonormal basis \( \{ 1, z, z^2, z^3, \ldots \} \) of \( H^2 \), one gets

\[
\| C_{\psi_n} - C_\psi \|_{\text{HS}}^2 = \int_T \left( \frac{1}{1 - |\psi_n|^2} + \frac{1}{1 - |\psi|^2} - 2 \Re \left\{ \frac{1}{1 - \overline{\psi_n} \psi} \right\} \right) dm
\]

\[
= \int_T \frac{1}{1 - |\psi_n|^2} dm + \int_T \frac{1}{1 - |\psi|^2} dm - 2 \Re \int_T \frac{1}{1 - \overline{\psi_n} \psi} dm.
\]

(9)

By hypothesis,

\[
\int_T \frac{dm}{1 - |\psi_n|^2} \to \int_T \frac{dm}{1 - |\psi|^2}.
\]

Since

\[
\left| \frac{1}{1 - \overline{\psi_n} \psi} \right| \leq \frac{1}{1 - |\psi|^2}, \quad m\text{-a.e.},
\]

the a.e. convergence hypothesis and the dominated convergence theorem combine to show that

\[
\int_T \frac{1}{1 - \overline{\psi_n} \psi} dm \to \int_T \frac{1}{1 - |\psi|^2} dm.
\]

By (9), it follows that \( \| C_{\psi_n} - C_\psi \|_{\text{HS}} \to 0 \). \( \square \)
The lemma above is an improvement in the framework of $H^2$ of [6, Theorem 4.5] and allows us to prove the main result of this section.

**Theorem 1.** If $C_\varphi$ is a Hilbert–Schmidt operator, then $\|C_{\varphi_n} - C_\varphi\|_{HS} \to 0$ if and only if $\|C_{\varphi_n}\|_{HS} \to \|C_\varphi\|_{HS}$ and $\|\varphi_n - \varphi\|_2 \to 0$.

**Proof.** First note that if $\|C_{\varphi_n} - C_\varphi\|_{HS} \to 0$, then clearly $\|C_{\varphi_n}\|_{HS} \to \|C_\varphi\|_{HS}$ and $\|\varphi_n - \varphi\|_2 \to 0$.

Conversely, assume by way of contradiction that $\|C_{\varphi_n}\|_{HS} \to \|C_\varphi\|_{HS} < \infty$ and $\|\varphi_n - \varphi\|_2 \to 0$, but $\|C_{\varphi_n} - C_\varphi\|_{HS} \not\to 0$. Then one can find some $\epsilon_0 > 0$ and a subsequence $\{\varphi_{n_k}\}$ such that $\|C_{\varphi_{n_k}} - C_\varphi\|_{HS} \geq \epsilon_0$. (10)

Since $\|\varphi_{n_k} - \varphi\|_2 \to 0$, there is a subsequence of $\{\varphi_{n_k}\}$ that converges a.e. to $\varphi$. Applying Lemma 1 to that subsequence, one gets a contradiction to (10). □

**Corollary 1.** Let $C_\varphi$ be a Hilbert–Schmidt composition operator and $\{\varphi_n\}$ a sequence in $S$. If

$$\int_T \frac{|\varphi_n - \varphi|}{(1 - |\varphi|)(1 - |\varphi_n|)} \, dm \to 0,$$

then $\|C_{\varphi_n} - C_\varphi\|_{HS} \to 0$.

**Proof.** Clearly

$$\|\varphi_n - \varphi\|_1 \leq \int_T \frac{|\varphi_n - \varphi|}{(1 - |\varphi|)(1 - |\varphi_n|)} \, dm \quad n = 1, 2, 3, \ldots,$$

so $\|\varphi_n - \varphi\|_2 \to 0$, since $\|\varphi_n - \varphi\|_2^2 \leq 2\|\varphi_n - \varphi\|_1$. Also

$$\left|\|C_{\varphi_n}\|_{HS}^2 - \|C_\varphi\|_{HS}^2\right| = \left| \int_T \frac{|\varphi_n|^2 - |\varphi|^2}{(1 - |\varphi|^2)(1 - |\varphi_n|^2)} \, dm \right|$$

$$\leq 2 \int_T \frac{|\varphi_n - \varphi|}{(1 - |\varphi|)(1 - |\varphi_n|)} \, dm \to 0. \quad \square$$

As another application, we prove the following “dominated convergence principle” for Hilbert–Schmidt norm-convergence of composition operators, which improves a result in [6], where uniform convergence is proved under assumptions slightly more restrictive that the ones below.

**Theorem 2.** Let $\varphi, \varphi_n \in S$, $n = 1, 2, \ldots$. If there exists a measurable function $\chi : \mathbb{T} \to [0, \infty]$ so that, for each $n$

$$|\varphi_n| \leq \chi \leq 1, \quad m\text{-a.e.}, \quad \int_T \frac{dm}{1 - |\chi|^2} < \infty,$$
and \(\|\varphi_n - \varphi\|_2 \to 0\), then \(\|C_{\varphi_n} - C_\varphi\|_{\text{HS}} \to 0\).

**Proof.** Observe that

\[
\frac{1}{1 - |\varphi_n|^2} \leq \frac{1}{1 - |\chi|^2}, \quad m\text{-a.e., } n = 1, 2, \ldots
\]

Since \(\{\varphi_n\}\) has a subsequence \(\{\varphi_{n_k}\}\) that converges a.e. to \(\varphi\), Lebesgue’s dominated convergence theorem leads to

\[
\|C_{\varphi_{n_k}}\|_{\text{HS}} \to \|C_\varphi\|_{\text{HS}} \leq \int_T \frac{dm}{1 - |\chi|^2} < \infty.
\]

Hence, by Theorem 1, \(\|C_{\varphi_{n_k}} - C_\varphi\|_{\text{HS}} \to 0\). Based on that, one can prove by contradiction that \(\|C_{\varphi_n} - C_\varphi\|_{\text{HS}} \to 0\) exactly as in the proof of that theorem.

As a corollary, we obtain Theorem 1 of [5]:

**Corollary 2.** The map \(\varphi \to C_\varphi\) is continuous from the open unit ball of \(H^\infty\) endowed with \(\|\|_\infty\) into the set of Hilbert–Schmidt composition operators.

**Proof.** Indeed, one may choose a positive constant \(r\) so that \(\|\varphi\|_\infty + r < 1\), set \(\chi := \|\varphi\|_\infty + r\), and apply the previous theorem (which is possible, since \(\|\varphi_n - \varphi\|_\infty \to 0\) implies that \(\|\varphi_n\|_\infty \leq \|\varphi\|_\infty + r\) for all values of \(n\) large enough).

Actually, in [5], the statement above is deduced as a consequence of the fact that the map \(\varphi \to C_\varphi\) is Lipschitz continuous from each ball of \(H^\infty\) of radius \(r\), \(0 < r < 1\), endowed with \(\|\|_\infty\) into the set of Hilbert–Schmidt composition operators (a fact the authors of [5] establish). As a last remark in this section, we would like to note that a simple upper norm estimate for the Hilbert–Schmidt norm of a difference of two composition operators proves that the map \(\varphi \to C_\varphi\) is Lipschitz continuous on subsets of the unit ball of \(H^\infty\) larger than the balls above.

**Remark 2.** For any pair of distinct symbols \(\varphi, \psi \in S\), let \(\chi := \max\{|\varphi|, |\psi|\}\). The following upper estimate of \(\|C_\varphi - C_\psi\|_{\text{HS}}\) holds:

\[
\|C_\varphi - C_\psi\|_{\text{HS}} \leq \sqrt{\int_T \frac{1 + \chi^2}{(1 - \chi^2)^3} \, dm \, \|\varphi - \psi\|_\infty}.
\] (12)

Hence for each \(R > 0\), the map \(\varphi \to C_\varphi\) is Lipschitz continuous on \(S_R := \{\varphi \in S: \int_T dm/(1 - |\varphi|)^3 \leq R\}\), that is there is some \(M > 0\) such that

\[
\|C_\varphi - C_\psi\|_{\text{HS}} \leq M \|\varphi - \psi\|_\infty, \quad \varphi, \psi \in S_R.
\]

**Proof.** By [3, p. 339], one can write
Proof. Assume first that \( \varphi \in S \) has a fixed point \( w \in U \) and is not an inner function. Recall that an inner function is an analytic selfmap of \( U \) whose radial limit-function is unimodular \( m \)-a.e. on \( T \).

For each \( w \in U \), \( C_w \) denotes the composition operator of constant symbol \( w \). Denote \( \varphi^{[n]} = \varphi \circ \cdots \circ \varphi \), \( n \) times for each \( n = 1, 2, \ldots \). Clearly \( C_{\varphi^{[n]}} = C_{\varphi}^n \).

**Theorem 3.** Let \( \varphi \in S \) be a non-inner symbol. If for some \( w \in U \), \( \varphi(w) = w \), then \( \|C_{\varphi}^n - C_w\| \to 0 \).

**Proof.** Assume first that \( w = 0 \). Let \( H_0^2 = \{ f \in H^2 : f(0) = 0 \} \). Recall that \( \|C_{\varphi}|H_0^2\| = \delta < 1 \) [8]. Consider any \( f \in H^2 \), \( \|f\|_2 = 1 \), and note that

\[
\|C_{\varphi}f - C_0f\|_2 = \|C_{\varphi}(f - f(0))\|_2 \leq \|f\| \cdot \|f(0)\|_2.
\]

Hence

\[
\|C_{\varphi}^n f - C_0 f\|_2 = \|C_{\varphi}(f \circ \varphi^{[n-1]} - f(0))\|_2 \leq \|f\| \cdot \|f^{[n-1]} - f(0)\|_2
\]

Iterating, one gets \( \|C_{\varphi}^n - C_0\| \leq \delta^n \to 0 \). A conformal conjugation argument takes care of the case \( w \neq 0 \). Indeed, consider the self-inverse conformal automorphism \( \alpha_w(z) = (w - z)/(1 - \bar{w}z) \), and set \( \psi = \alpha_w \circ \varphi \circ \alpha_w \). Note that \( \psi(0) = 0 \), hence \( \|C_{\varphi}^n - C_{\psi}^n\| \to 0 \) if \( m, n \to \infty \). This fact implies \( \|C_{\varphi}^m - C_{\psi}^m\| \to 0 \) if \( m, n \to \infty \). Indeed, for each \( k \) one has

\[
C_{\varphi}^k = C_{\alpha_w} C_{\psi}^k C_{\alpha_w} \text{ and hence } \|C_{\varphi}^m - C_{\psi}^n\| \leq \|C_{\alpha_w}\|^2 \|C_{\psi}^m - C_{\psi}^n\|, m, n = 1, 2, \ldots.
\]

We established that the sequence \( \{C_{\varphi}^n\} \) is norm-convergent. Let \( T \) denote its limit. It is well known that \( \varphi^{[n]} \to w \) uniformly on compact subsets of \( U \), hence also weakly in \( H^2 \) (see [7, the Denjoy–Wolff Theorem]). Since \( C_{\varphi}(z) = \varphi^{[0]} \), it follows by Remark 1 that \( T = C_w \).

The argument used to prove Theorem 3 occurs, with minor changes, in [2]. There it is used to show that the iterates of \( \varphi \) converge to \( w \) in the \( H^2 \) norm. We included the proof of Theorem 3 for the sake of completeness.

If \( \varphi \) is inner, then for any \( n : \|C_{\varphi}^n - C_w\| \geq \|C_{\varphi}^n - C_w(z)\| \geq 1 - |w| \), so that Theorem 3 cannot be extended to this case. In fact, by a result of Berkson ([1], see also [9]) \( \|C_{\varphi}^n - C_w\| \geq 1 \) in this case.

The situation when \( \|C_{\varphi}^n - C_w\|_{HS} \to 0 \) is characterized in the following theorem.
Theorem 4. Let $\varphi$ be a non-inner function with a fixed point $w$ in $U$. Then $\|C_n^\varphi - C_w\|_{HS} \to 0$ if and only if there is some positive integer $k$ such that $C_k^\varphi$ is a Hilbert–Schmidt operator.

Proof. The necessity is evident, given that obviously $C_w$ is Hilbert–Schmidt. To prove the sufficiency, assume first that $w = 0$. Note that $\psi^{[n]} \to 0$, $m$-a.e. Indeed, using the notation in the proof of Theorem 3, observe that, in that proof, we obtained that 

$$\sum_{n=1}^{\infty} \| (C_n^\varphi - C_0) (z) \|^2 < \infty,$$

hence

$$\sum_{n=1}^{\infty} \| \psi^{[n]} \|^2 < \infty,$$

so, by Lebesgue’s monotone convergence theorem,

$$\int_T \left( \sum_{n=1}^{\infty} |\psi^{[n]}|^2 \right) dm < \infty,$$

which implies $\psi^{[n]} \to 0$, $m$-a.e.

Now, by the Schwarz lemma in classical complex analysis,

$$|\psi^{[n]}| \leq |\psi^{[k]}|, \quad m$-a.e., $n \geq k,$

so, setting $\chi := |\psi^{[k]}|$ in Theorem 2 leads to the desired conclusion when $w = 0$. A standard conformal conjugation argument takes care of the general case like in the proof of Theorem 3. Indeed, for $w$ arbitrary, one can associate to $\varphi$ the conformal conjugate $\psi$ as in that proof and note that $\|C_m - C_n\|_{HS} \leq \|C_m - N\|_{HS} \leq \|C_m - C_n\|_{HS}, m, n = 1, 2, \ldots$, by [4, p. 1012, Corollary 5]. Thus, by the first part of this proof, the sequence $\{C_{\psi_n}\}$ tends to an operator $T$ in the Hilbert–Schmidt norm. One shows that $T = C_w$ exactly as in the proof of Theorem 3. 

In the argument above we needed the fact that, if $\varphi$ fixes a point $w$ in $U$ and is not an inner function, then its iterates tend a.e. to $w$. This was first established in [2]. For the sake of the self-sufficiency of the current paper, we decided to include the proof, rather than just refer the reader to [2].

The situation when the assumptions in Theorem 3 hold but those in Theorem 4 do not, may occur, as we show in the following.

Example 1. Let $\varphi(z) = (z^3 + 1)/2$. This symbol satisfies the assumptions in Theorem 3, hence there is $w \in U$ so that $\|C_n^\varphi - C_w\| \to 0$, but $\|C_n^\varphi - C_w\|_{HS} \not\to 0$.

Proof. Clearly $\varphi$ is not inner. Indeed, by the triangle inequality, $|z^3 + 1| \leq 1,$ for all $z$ in the closed unit disk, and equality occurs only if $z$ is a cube root of 1. The fixed points of
ϕ are the zeros of \(z^3 - 2z + 1\), a polynomial that is real on the real line, positive at .5 and negative at .7. Therefore, ϕ has a fixed point \(w \in \mathbb{U}\) and hence satisfies the assumptions in Theorem 3. On the other hand, all the iterates of ϕ have finite angular derivatives at 1. Thus, \(C_\varphi^n\) is not compact, \(n = 1, 2, \ldots\), (see [3] or [7]), and hence, \(C_\varphi^n\) cannot be Hilbert–Schmidt. □

By Remark 1, the power-sequence \(\{C_\varphi^n\}\) of a composition operator with symbol without fixed points in \(\mathbb{U}\) is weakly divergent, since in that case, there is a unimodular constant function \(\omega\) toward which \(\{\varphi^n\}\) tends weakly, (by the Denjoy–Wolff Theorem [3,7]).

4. Uniform convergence

In this section we establish an upper norm estimate for the norm of a difference of two composition operators and show that if condition (11) holds, but one drops the requirement that \(C_\varphi\) be Hilbert–Schmidt, one can still prove that \(\|C_\varphi^n - C_\varphi\| \to 0\). We begin with the norm estimate.

**Theorem 5.** For any \(\varphi, \psi \in \mathcal{S}\) the following inequality holds:

\[
\|C_\varphi - C_\psi\| \leq 2 \sqrt{\int_T \frac{|\varphi(u) - \psi(u)|}{(1 - |\varphi(u)|)(1 - |\psi(u)|)} \, dm(u)}.
\] (13)

**Proof.** First we prove a simple inequality involving the usual Poisson kernel \(P(z, \zeta)\), \(z \in \mathbb{U}\), \(\zeta \in \mathbb{T}\), namely

\[
|P(z, \zeta) - P(w, \zeta)| \leq 2 \frac{|z - w|}{|\zeta - z||\zeta - w|}, \quad z, w \in \mathbb{U}, \ \zeta \in \mathbb{T}.
\]

Indeed,

\[
|P(z, \zeta) - P(w, \zeta)| = \left| \text{Re} \left( \frac{\zeta + z}{\zeta - z} - \frac{\zeta + w}{\zeta - w} \right) \right| \leq \left| \frac{\zeta + z}{\zeta - z} - \frac{\zeta + w}{\zeta - w} \right| = 2 \frac{|z - w|}{|\zeta - z||\zeta - w|}.
\]

Next, note that the above inequality can be used to show that

\[
|f(z) - f(w)|^2 \leq 4|z - w| \sup_{\zeta \in \mathbb{T}} \left( \frac{1}{|\zeta - z||\zeta - w|} \right) \|f\|_2^2
\]

\[
\leq \frac{4|z - w||f\|_2^2}{(1 - |z|)(1 - |w|)}, \quad z, w \in \mathbb{U}, \ f \in H^2.
\] (14)

Indeed, using the Cauchy–Schwartz inequality,

\[
|f(z) - f(w)|^2 \leq \left( \int_T |P(z, \zeta) - P(w, \zeta)| \|f(\zeta)\| \, dm(\zeta) \right)^2
\]
\[
\begin{align*}
&\leq \int_T |P(z, \zeta) - P(w, \zeta)|^2 \, dm(\zeta) \| f \|^2_2 \\
&\leq \sup_{\zeta \in T} |P(z, \zeta) - P(w, \zeta)| \int_T |P(z, \zeta) - P(w, \zeta)| \, dm(\zeta) \| f \|^2_2 \\
&\leq 4|z - w| \sup_{\zeta \in T} \left( \frac{1}{|z - \zeta||\zeta - w|} \right) \| f \|^2_2 \\
&\leq \frac{4|z - w| \| f \|^2_2}{(1 - |z|)(1 - |w|)}, \quad z, w \in \mathbb{U}, \ f \in \mathcal{H}^2.
\end{align*}
\]

Substitute \( z \) by \( \varphi(u) \), \( w \) by \( \psi(u) \), and integrate \( dm(u) \) to obtain (13). \( \square \)

**Corollary 3.** If condition (11) holds, then \( \| C_{\varphi_n} - C_{\varphi} \| \to 0. \)

Clearly, inequality (13) is interesting only if \( \varphi \neq \psi \), \( |\varphi| < 1 \), and \( |\psi| < 1 \), \( m\)-a.e. Indeed, the integral involved in it is infinite if \( \varphi \neq \psi \) and any of these functions has unimodular radial function on a measurable subset of \( T \) having positive measure.

The paper [9] contains an upper norm-estimate for the difference of two composition operators. The methods used in [9, Theorem 3.2] can be adapted to show that, if the integral in estimate (13) is finite, then the operator \( C_{\varphi} - C_{\psi} \) must be compact.

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**References**