

MATH 4760/8766

Homework Assignment # 1: Metric Spaces

Problem 1: Show that the following are all metrics in the space $\mathbf{X} = \mathbb{R}$:

(a) $d(x, y) = |x - y|$

(b) $d(x, y) = 5 \cdot |x - y|$

(c) $d(x, y) = |x^3 - y^3|$

Show that $d(x, y) = |xy|$ does **not** define a metric in this space.

Problem 2: Show that the following are metrics in the space $\mathbf{X} = \mathbb{R}^2$:

(a) $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ (Euclidean metric)

(b) $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$ (Manhattan metric)

(c) $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ if the points x, y and $O = (0, 0)$ are collinear, and $d(x, y) = \sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2}$ otherwise (metric Center).

Problem 3: *The code space on N symbols.*

Let N be a natural number greater or equal to 2 and let Σ_N be the set of all infinite words (sequences)

$$x = x_1x_2x_3x_4x_5x_6 \dots \quad \text{where each } x_i \in \{0, 1, 2, \dots, N - 1\}.$$

(Elements of Σ_N may be called *codes*.) For two codes $x = x_1x_2x_3x_4 \dots \in \Sigma_N$ and $y = y_1y_2y_3y_4 \dots \in \Sigma_N$ we let

$$d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{(N + 1)^i}$$

Show that d is a function from $\Sigma_N \times \Sigma_N$ into \mathbb{R} (i.e. the series above always converges to a finite number) and that (Σ_N, d) is a metric space.

Problem 4: Two metrics d_1 and d_2 on a space \mathbf{X} are *equivalent* if there exist constants $0 < c_1 < c_2 < \infty$ such that

$$(\forall (x, y) \in \mathbf{X} \times \mathbf{X}) (c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y)).$$

Show that if metrics d_1, d_2 on space \mathbf{X} are equivalent, then there are constants $0 < e_1 < e_2 < \infty$ so that

$$(\forall (x, y) \in \mathbf{X} \times \mathbf{X}) (e_1 d_2(x, y) \leq d_1(x, y) \leq e_2 d_2(x, y)).$$

Problem 5: Are the metrics listed in Problems 1, 2 equivalent?

Problem 6: Define two metrics on the space $\mathbf{X} = (0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$ by

$$d_1(x, y) = |x - y| \quad \text{and} \quad d_2(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right|.$$

Show that (\mathbf{X}, d_1) and (\mathbf{X}, d_2) are not equivalent metric spaces.