PROBLEM 3* Evaluate the integral

$$\int_0^{\pi/3} x \tan^2(x)\, dx$$

by using a [illegible] integration by parts.
(Show all your work - present your solution fully.)

We will first find the indefinite integral $\int x \tan^2(x)\, dx$.

We note that $\int \tan^2(x)\, dx = \int \frac{\sin^2(x)}{\cos^2(x)}\, dx = \int \frac{1-\cos^2(x)}{\cos^2(x)}\, dx$ so

$$\int \tan^2(x)\, dx = \int \frac{1}{\cos^2(x)} - 1\, dx = \int \frac{1}{\cos^2(x)}\, dx - \int dx = \tan(x) - x + C.$$

Therefore

$$\int x \tan^2(x)\, dx = \int x \cdot (\tan(x) - x)'\, dx \underset{\text{by parts}}{\overset{\text{integration}}{=}}$$

$$x \cdot (\tan(x) - x) - \int (x)' (\tan(x) - x)\, dx =$$

$$x \tan(x) - x^2 - \int \tan(x)\, dx + \int x\, dx.$$

Since $\int x\, dx = \frac{1}{2}x^2 + C$ and

$$\int \tan(x)\, dx = \int \frac{\sin(x)}{\cos(x)}\, dx \underset{\substack{u = \cos(x) \\ du = -\sin(x)dx}}{\overset{\text{substitution}}{=}} -\int \frac{du}{u} =$$

$$-\ln(|u|) + C = -\ln(|\cos(x)|) + C,$$

we get

$$\int x \tan^2(x)\, dx = x\tan(x) - x^2 + \ln(|\cos(x)|) + \frac{1}{2}x^2 + C =$$
$$x\tan(x) - \frac{1}{2}x^2 + \ln(|\cos(x)|) + C.$$

Now we use the fundamental theorem of calculus to get

$$\int_0^{\pi/3} x \tan^2(x)\, dx = \left(x\tan(x) - \frac{1}{2}x^2 + \ln(\cos(x))\right)\Big]_0^{\pi/3} =$$

$$\frac{\pi}{3} \cdot \sqrt{3} - \frac{1}{2}\pi^2 \cdot \frac{1}{9} + \ln\left(\cos\left(\frac{\pi}{3}\right)\right) - 0 = \pi\frac{\sqrt{3}}{3} - \frac{\pi^2}{18} + \ln\left(\frac{1}{2}\right) =$$

$$\pi\frac{\sqrt{3}}{3} - \frac{\pi^2}{18} - \ln(2)$$