# SHELAH'S SEARCH FOR PROPERNESS FOR ITERATIONS WITH UNCOUNTABLE SUPPORTS

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Abstract. Soon after the notion of "proper forcing" was formalized, it became clear that there are no direct generalizations of "proper forcing" which would work for iterations with uncountable supports. However several steps towards preservation theorems for iterations with uncountable supports have been already done and we will review some of those results. The lucky numbers here are 587, 655, 667, 777, 860, 890.

## 1. A bit of History: CS iterations and Proper Forcing

In March 2003, when visiting Saharon in Jerusalem, I asked him

> *Do you remember where you came with the definition of proper forcing first? And when was that?*

As usual, the answer that I received was more than I was really looking for, but, I believe, it is very interesting. What Saharon told me was basically this:

*Till Summer 1976 I have not considered myself a true set theorist (but certainly I have considered myself a model theorist). I have done combinatorial set theory and I have done some easy forcing arguments, like*

- *adding supercompact–many Cohen reals and its effect on the existence of $\kappa$–free not $\kappa^+$–free groups (see Shai Ben-David* [Bd:C1]*),*
- *adding one Cohen real over a model of CH makes some partition relation on graphs of size $\aleph_1$ true; A. Hajnal and P. Komjath have continued this, see their dual to* [Sh 289].

*I have the vague impression that the logicians in Berkeley did not appreciate this too much and I think that they considered forcing as the thing in math logic. M. Magidor was there and he has not noticed this (or perhaps he forgot this), but when I visited Berkeley January–June 1978 Leo Harrington certainly was (kindly) sure that the leaders of set theory are much better than the leaders in other brunches of logic. (In our friendly argument we arrived to the question if there more set theorist or model theorist, so we played the Cantor game: we start naming people and the one without a new name should concede defeat. We advanced nicely till I arrived to East Germany and he was lost feeling this is not really fair play. He remembered it well enough to tell Udi Hrushovski and sometime he told me that maybe now we should play the game interchanging our roles (of course he really is a recursion theorist...))*

*Probably in March'72 I thought that from the proof of Jensen and Johnsbråten of the consistency of GCH + SH we can deduce an axiom. Jensen did not answer, so I have written Fenstad (complaining on Jensen ) and got it. I knew I did not*

*Date*: October 2005.

*know forcing, so I went to Levy, but he was dean.... So as Uri Abraham has come to him to do MSc, he sent him to me to hear the problem and he ended up doing his thesis with me. It become a paper with Devlin who was working on it* [ADSh 81]*, but does not seem to me realization of the original aim.*

*After some people were claiming that they can improve my "diamond on every stationary subset of* $\aleph_1$ *implies every Whitehead group in* $\aleph_1$ *is free" from CH, they received serious treatment: I was motivated to try to prove it is necessary.*

*In Summer 1976 I understood things better: the weak diamond first (Devlin started by proving for Abelian groups), then* [Sh 64]*. Writing it I have really wrote what is an iteration and I understood it; this is not worthless now – generalizations to higher cardinals are sometimes similar to it. Though the motivation was from Group Theory, it solved the following set theoretic problem: can you have diamond on one stationary subset of* $\omega_1$ *but not another?*

*I have also done some more forcing work with Uri Abraham* [AbSh 102]*; in this period another iteration is* [Sh 80]*, this is most relevant to larger cardinals, but then I found out that Baumgartner had unpublished work – not the same but close (he did not publish as Laver...). Another forcing paper (but no iteration) is* [Sh 76].

*Then in Fall 1977 I was in Madison, so naturally I was interested in General Topology: Archangelski problem (see* [Sh:E3]*, I forgot to publish it but it has been well known) is just special forcing but the proof of the no P–point (take product and a subforcing of it) was the next step.*

*Then later, probably in November, as I have proofread* [Sh:a] *I thought if the many non-isomorphic models results are best possible. There is a notice in the* Notices of AMS *of the two main results of* [Sh 100]*. In this respect the first proofs were by taking in limits of countable cofinality not the full limit but large enough subset, first with the ccc (like* [Sh 669]*), then without it, but I had to finish proofreading* [Sh:a]*, so I delayed the writing.*

*This I did in February 1978, I used more naturally the full inverse limit; I then (still February'78) found a neater proof in my eye, which is represented in* [Sh:b, Chp 9] *– free limit.*

*In Berkeley in the Winter Trimester [1978], I taught Model Theory and in the second I taught proper forcing - including Con(CH+SH). At the time Wimmers was supposed to write up the proper forcing - I suggest him instead of writing the proof as appeared, the proof as in* [Sh:b, Chp 5]*, but after a year or two, as he did not do, I reminded him maybe somewhat strongly and he answered that he was returning it to me.*

*Meanwhile, I understood the Berkeley people prefer cardinal arithmetic and continued my* [Sh 71]*, I dealt with the referee report. Then I worked on* [Sh 111] *which has a not so good history with referees (see "the green grass syndrome" in the* Future[1]*)*

[1] In [Sh:E16] Shelah wrote:
*But there is more to it, less cynically. I have the weak "neighbor's grass is greener" syndrome. In the strong sense you "know" your neighbor's grass is greener. I "know" it is not, but I would like to have a proof; also some doubts always linger as I have some cocky neighbors. So I was curious to try my hand on descriptive set theory too, for example. The reader may ask: how do I like my neighbor's grass? Usually the grass is not only quite green, but interesting; but certainly not more so.*

Based on the history as described by Shelah himself, I would like to risk the claim that at the early stages of *Proper Forcing* the primary motivation was coming from the problems related to *not adding reals*. It was somewhat later that the properness was matched with additional properties like "${}^{\omega}\omega$–bounding", "the PP–property"", "preserving P-points" (see Blass and Shelah [BsSh 242]) or "$(f,g)$–bounding" (see [Sh 326]) while adding reals. In the end of the 1980s, majority of the proper forcing users forced to *add* reals with preserving some specific properties of the ground model reals. In the 1990s, one of the ultimate examples of the success of the *proper forcing* were, arguably, the preservation theorems of [Sh:f, Ch VI, XVIII] and their simplified and cleaned versions presented e.g. in Bartoszyński and Judah [BaJu95]. One of the reasons that in 2000 Mathematics Subject Classification we have

`03E17 Cardinal characteristics of the continuum`

is the plethora of independence results obtained by the means of CS iterations of proper forcing notions.

## 2. So what about iterations with uncountable supports?

While there is still a lot of open problems left in the theory of forcing iterated with finite and/or countable support and we still need to expand our preservation theorems, there is a sense that we *understand these iterations pretty well.* Therefore we may look at other iterations and ask for parallel theorems.

2.1. **Why not "straightforward"?** The first attempt could be to do nothing special and just repeat what we have done for CS iterations. We could start in the way suggested in [Sh 100] already (but not used there).

**Definition 2.1.** Let $\lambda = \lambda^{<\lambda}$. A notion of forcing $\mathbb{P}$ is said to be $\lambda$–proper if for all sufficiently large regular cardinals $\chi$, there is some $x \in \mathcal{H}(\chi)$ such that whenever $M$ is an elementary submodel of $\mathcal{H}(\chi)$ satisfying

$$|M| = \lambda, \qquad \mathbb{P}, x \in M \qquad M^{<\lambda} \subseteq M$$

and $p$ is an element of $M \cap \mathbb{P}$, there is a condition $q \geq p$ such that

$$q \Vdash \text{“}M[\underset{\sim}{G}_P] \cap \mathrm{Ord} = M \cap \mathrm{Ord}\text{”}.$$

The $\lambda$–properness may seem to be a straightforward generalization of "proper", with the right consequences in place. For instance:

**Proposition 2.2** (Folklore; Hyttinen and Rautila [HyRa01, Section 3])**.** *Assume $\lambda^{<\lambda} = \lambda$ is an uncountable cardinal.*

1. *If a forcing notion $\mathbb{P}$ is either strategically $(<\lambda^+)$–complete or it satisfies the $\lambda^+$–chain condition, then $\mathbb{P}$ is $\lambda$–proper.*
2. *If $\mathbb{P}$ is $\lambda$–proper, $p \in \mathbb{P}$, $\underset{\sim}{A}$ is a $\mathbb{P}$–name for a set of ordinals and $p \Vdash |\underset{\sim}{A}| \leq \lambda$, then there are a condition $q \in \mathbb{P}$ stronger than $p$ and a set $B$ of size $\lambda$ such that $q \Vdash \underset{\sim}{A} \subseteq B$.*
3. *If $\mathbb{P}$ is $\lambda$–proper, then*

$$\Vdash_{\mathbb{P}} \text{“ } (\lambda^+)^{\mathbf{V}} \textit{ is a regular cardinal ”}.$$

*Moreover, forcing with $\mathbb{P}$ preserves stationary subsets of $\lambda^+$.*

Also chain condition results look similarly:

**Proposition 2.3** (Folklore; Eisworth [Ei03, Prop 3.1])**.** *Assume* $\lambda^{<\lambda}=\lambda$, $2^\lambda=\lambda^+$, *and let* $\bar{\mathbb{P}}=\langle \mathbb{P}_i,\underset{\sim}{\mathbb{Q}}_i : i<\lambda^{++}\rangle$ *be a* $\lambda$*–support iteration such that*

(1) $\mathbb{P}_i$ *is* $\lambda$*–proper for* $i\leq\lambda^{++}$
(2) $\Vdash_{\mathbb{P}_i}$ " $|\underset{\sim}{\mathbb{Q}}_i|\leq\lambda^+$ "

*Then* $\mathbb{P}_{\lambda^{++}}$ *satisfies the* $\lambda^{++}$*–chain condition.*

What is missing? The main point of properness is the preservation theorem for CS iterations. If one tries to repeat the proof of the preservation theorem for $\lambda$–support iterations of $\lambda$–proper forcing notions, then one faces difficulties at limit stages of cofinality less than $\lambda$ caused by the fact that it is inconvenient to diagonalize $\lambda$ objects in less than $\lambda$ steps. This is a more serious obstacle than just a technicality. Let us consider the following forcing notions.

**Example 2.4** (Shelah, [Sh:b, Appendix])**.** Assume that $\lambda=\lambda^{<\lambda}$ is an uncountable cardinal and let $\mathcal{S}^{\lambda^+}_\lambda\stackrel{\rm def}{=}\{\delta<\lambda^+ : {\rm cf}(\delta)=\lambda\}$. Suppose that a sequence $\langle A_\delta,h_\delta : \delta\in\mathcal{S}^{\lambda^+}_\lambda\rangle$ is such that for each $\delta\in\mathcal{S}^{\lambda^+}_\lambda$:

(a) $A_\delta\subseteq\delta$, ${\rm otp}(A_\delta)=\lambda$ and $A_\delta$ is a club of $\delta$, and
(b) $h_\delta : A_\delta\longrightarrow 2$.

The forcing notion $\mathbb{Q}^*=\mathbb{Q}^*(\langle A_\delta,h_\delta:\delta\in\mathcal{S}^{\lambda^+}_\lambda\rangle)$ is defined as follows:
**a condition in** $\mathbb{Q}^*$ is a tuple $p=(u^p,v^p,\bar{e}^p,h^p)$ such that

(a) $u^p\in[\lambda^+]^{<\lambda}$, $v^p\in[\mathcal{S}^{\lambda^+}_\lambda]^{<\lambda}$,
(b) $\bar{e}^p=\langle e^p_\delta : \delta\in v^p\rangle$, where each $e^p_\delta$ is a closed bounded subset of $A_\delta$, and $e^p_\delta\subseteq u^p$, and
(c) if $\delta_1<\delta_2$ are from $v^p$, then

$$\sup(e_{\delta_2})>\delta_1 \quad\text{ and }\quad \sup(e_{\delta_1})>\sup(A_{\delta_2}\cap\delta_1),$$

(d) $h^p : u^p\longrightarrow\lambda$ is such that for each $\delta\in v^p$ we have that $h^p\restriction e_\delta\subseteq h_\delta$.

**The order** $\leq$ **of** $\mathbb{Q}^*$ is such that $p\leq q$ if and only if $u^p\subseteq u^q$, $h^p\subseteq h^q$, $v^p\subseteq v^q$, and for each $\delta\in v^p$ the set $e^q_\delta$ is an end-extension of $e^p_\delta$.

Plainly, under the assumptions of 2.4, the following holds true.

**Proposition 2.5.** *The forcing notion* $\mathbb{Q}^*$ *is* $(<\lambda)$*–complete and* $|\mathbb{Q}^*|=\lambda^+$*. It satisfies the* $\lambda^+$*–chain condition, so it is also* $\lambda$*–proper.*

If $\lambda=\lambda^{<\lambda}$ is not inaccessible and $2^\lambda=\lambda^+$, then some $\lambda$–support iterations of forcing notions like $\mathbb{Q}^*$ are not $\lambda$–proper, as a matter of fact this bad effect happens quite often. The problem comes from the fact that such iterations are $(<\lambda)$–complete and if they were $\lambda$–proper, they would also have satisfied $\lambda^{++}$–cc. Therefore by a suitable bookkeeping device and iterated forcing we could build a forcing notion forcing "$\lambda=\lambda^{<\lambda}$ is not inaccessible and club uniformization for continuous lader systems holds true". However, this is not possible:

**Theorem 2.6** (Shelah, [Sh:b], [Sh:f, App 3.6(2)])**.** *Assume* $\theta<\lambda={\rm cf}(\lambda)$, $2^\theta=2^{<\lambda}=\lambda$*. Furthermore suppose that for each* $\delta\in\mathcal{S}^{\lambda^+}_\lambda$ *we have a club* $A_\delta$ *of* $\delta$*. Then we can find a sequence* $\langle d_\delta:\delta\in\mathcal{S}^{\lambda^+}_\lambda\rangle$ *of colourings such that*

- $d_\delta : a_\delta\longrightarrow 2$ *and*
- *for any* $h:\lambda^+\longrightarrow\{0,1\}$ *for stationarily many* $\delta\in\mathcal{S}^{\lambda^+}_\lambda$*, the set* $\{i\in A_\delta : d_\delta(i)\neq h(i)\}$ *is stationary in* $A_\delta$*.*

2.2. **Minimal General Program.** Thus we cannot have anything that would be a direct extension of the theory of "proper forcing and CS iterations" to the context of iterations with uncountable supports. Still, there are in the literature many examples of forcing iterated with uncountable supports, so we know that at least in some cases the limits of such iterations do not collapse too much. Therefore one hopes that in long run we will be able to find properties $\mathbf{P}^A_\lambda$ and $\mathbf{P}^B_\lambda$ of (strategically) $(<\lambda)$–complete forcing notions such that

- $\mathbf{P}^B_\lambda(\mathbb{P})$ implies that forcing with $\mathbb{P}$ does not collapse $\lambda^+$ (and perhaps even more),
- (the limit of) $\lambda$–support iteration of forcing notions with $\mathbf{P}^A_\lambda$ has the property $\mathbf{P}^B_\lambda$,
- all interesting forcing notions have the property $\mathbf{P}^A_\lambda$.

Of course, we would like to have *real preservation theorems,* but we can live without them. Why do we restrict ourselves to $(<\lambda)$–complete (or at least strategically $(<\lambda)$–complete) forcing notions? We want to work with $\lambda$–support iterations and $\lambda$–properness–like conditions guarantee that the limit of the iteration does not collapse $\lambda^+$, some finer chain condition arguments will hopefully take care of preserving larger cardinals (see, e.g., 2.3). But we also need some conditions helping us to preserve cardinals and cofinalities below $\lambda$ and demands like strategic $(<\lambda)$–completeness seem to be reasonable.

It may well be that requiring that one pair $(\mathbf{P}^A_\lambda, \mathbf{P}^B_\lambda)$ applies to *all* interesting forcing notions is too much. After all we could be quite happy with having several of such pairs, each applying to a resonably large class of forcings. But which forcing notions should we be interested in? The answer is, of course, closely related to the question which *problems* are most important to us.

## 3. "No new reals" again

Very important applications of CS iterations of forcing notions adding reals were given in the mid 1970s already (e.g., Laver's consistency of Borel Conjecture in [Lv76a]), but it seems that they were not the original inspiration for the developments of the general theory of such iterations. The problems around "CH with some consequences of MA", and iterated forcing without adding new reals were treated (by Shelah) more seriously at first. He showed similar approach to iterations with uncountable supports.

Considerable effort was given to investigating how far one can push the ZFC results on no uniformization, weak and other diamonds and likes. The paper that really started the line of Shelah's papers on "the theory of uncountable support iterations" is [Sh 186]. Several of the concepts here would look very familiar to the readers of the following works, most notably the main ingredients of the proof of the iteration theorem [Sh 186, Theorem 2.5]. Unfortunately, most of the proofs and arguments are left to the reader and the paper is a kind of difficult to follow. Maybe those factors contributed to apparent lack of interest in both the paper and this direction or research (the iteration theory). This situation was even stranger, as there was (and still is) a considerable interest in generalizations of MA to higher cardinals as well as in the Generalized Souslin Hypothesis and its relationship to GCH.

Some time in 1995, Shelah looked again at these problems. His main motivation was coming from joint projects with Göbel (like Göbel and Shelah [GbSh 579]) and questions like:

($\circledast$) is it consistent with ZFC + GCH that for some regular $\kappa$ there is an almost free Abelian group of cardinality $\kappa$, but every such Abelian group is a Whitehead one?

The ideas concerning the corresponding forcing results remained undeveloped and neglected for a year. In Fall 1995 went for a Sabbatical and he spent the period January — July 2006 at Uniwersity of Wisconsin, Madison. He lectured there on some of works in progress, and the second topic[2] he presented there was the solution to ($\circledast$). Madison lectures were a base for what eventually became [Sh 587].

**(Non-)paid commercial:**

*Shelah's* [Sh 587] *is a very nicely written paper, full of details and practically self-contained. Proofs there are presented in detail and the number of misprints etc relatively small. The paper was presented in a* Featured Review *in MathSciNet for good reasons: it is a* must read *for anybody interested in the theory of iterated forcing, but more importantly, everybody actually can read it!*

Shelah's answer to ($\circledast$) is (not so surprisingly) related to the uniformization property for colourings on ladder systems. Forcing to get the respective uniformizations is the main criterion for the choice of the setting and definitions. In [Sh 587] Shelah describes 5 main and 2 auxiliary cases coming from this context and he deals with the first two:

Case A: $\kappa = \lambda^+$, $\lambda = \lambda^{<\lambda}$, $S \subseteq S^\kappa_\lambda$, and both $S$ and $S^\kappa_\lambda \setminus S$ are stationary;
Case B: $\kappa$ is strongly inaccessible and $S \subseteq \kappa$ is stationary but "thin" in a sense.

(We want to have the uniformization property for all ladder systems on $S$.) While these two cases are different, author's approach to both of them has a similar flavour. In both cases the iteration is with $(<\kappa)$–support and in each case the author introduces two completeness properties: one "lives on" the complement of the set $S$, while the other property has more global character, and what is really used is a combination of the two properties. The first completeness property implies that no new bounded subsets of $\kappa$ are added, it is preserved in $(<\kappa)$–support iterations and it is enough to control the cardinal arithmetic in the forcing extensions by the iterations of canonical examples, but it is not enough to guarantee that the set $S$ (or its relatives) remain stationary. Only the combination of the two completeness properties guarantees that the stationarity of $S$ (and relatives) is preserved in $(<\kappa)$–support iterations. The iteration arguments are expansions of those from [Sh 186] and trees of conditions are used extensively here.

The next of the cases isolated in [Sh 587] is dealt with in [Sh 667, Section 2].

Case C: $\lambda$ is singular, $S \subseteq S^{\lambda^+}_{\mathrm{cf}(\lambda)}$ is a non–reflecting stationary set, $\kappa = \lambda^+$.

The definition, concepts and theorems for this case are modifications of those from [Sh 587, Case B] and it is clear that they were written *by the way of cannibalization* of [Sh 587]. Unfortunately, this process was not carried out properly and [Sh 667] is very difficult to follow.

[2]the first topic was what later became [Sh 594] — another forcing paper that I would recommend for reading

Summarizing, in [Sh 587] and [Sh 667] Shelah introduced several variants of *strong completeness/properness* and proved that they can be iterated (in some cases getting real preservation theorems). However, those results were tailored for forcing notions *not adding new bounded subsets* of $\kappa$ in $(<\kappa)$–support iterations, generalizing preservation of "$S$-complete proper" in CS iterations. *(So this is not what I really like.)*

## 4. The treasures that conquered my heart

A large part of my mathematical life was related to Set Theory of the Reals and Forcing for the Reals. So it should be no surprise that instead of the uniformization properties with GCH I would rather enjoy

4.1. **Set Theory of the $\lambda$–reals.** A number of cardinal characteristics related to the Baire space $\omega^\omega$, the Cantor space $2^\omega$ and/or the combinatorial structure of $[\omega]^\omega$ can be extended to the spaces $\lambda^\lambda, 2^\lambda$ and $[\lambda]^\lambda$ for any infinite cardinal $\lambda$. The process of "generalizing cardinal characteristics of the continuum" is very straightforward, though there are still some choices to be made. For instance, we may follow the pattern of replacing the quantifiers $(\forall^\infty n), (\exists^\infty n)$ by $(\exists j < \lambda)(\forall i > j)$, $(\forall j < \lambda)(\exists i > j)$ (respectively), or we may replace them by "on a club of $\lambda$", "on a stationary subset of $\lambda$" (respectively), etc. Also, for some cardinal characteristics of $\lambda^\lambda$ we may have relatives generated by looking at small families of objects (instead of single objects). So, for example, we have the following relatives of the dominating number $\mathfrak{d}$.

**Definition 4.1.**

- A family $\mathcal{F} \subseteq \lambda^\lambda$ is *a dominating family* if
$$(\forall h \in \lambda^\lambda)(\exists f \in \mathcal{F})(\forall j < \lambda)(h(j) < f(j)),$$
The dominating number $\mathfrak{d}_\lambda$ is
$$\mathfrak{d}_\lambda = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \lambda^\lambda \text{ is a dominating family }\}.$$
- If $D$ is a filter on $\lambda$, then replacing "$(\forall j < \lambda)$" by "$(\exists A \in D)(\forall j \in A)$" leads to definitions of a *$D$–dominating family* and the cardinal number $\mathfrak{d}^D_\lambda$.
- Let $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda\rangle$, $\mu_\alpha < \lambda$. Let $\mathcal{S}_{\bar{\mu}}$ be the family of all sequences $\bar{a} = \langle a_\alpha; \alpha < \lambda\rangle$ such that $a_\alpha \in [\lambda]^{<\mu_\alpha}$ (for all $\alpha < \lambda$). We define
$$c(\bar{\mu}) = \min\big\{|\mathcal{Y}| : \mathcal{Y} \subseteq \mathcal{S}_{\bar{\mu}} \ \&\ (\forall f \in {}^\lambda\lambda)(\exists \bar{a} \in \mathcal{Y})(\forall \alpha < \lambda)(f(\alpha) \in a_\alpha)\big\}.$$

Following the tradition of Set Theory of the reals we may call cardinal numbers defined this way for $\lambda^\lambda$ (and related spaces) *cardinal characteristics of $\lambda$–reals.* For the reasons described above, the menagerie of such characteristics seems to be much larger than the one for the continuum, and there has been already some interest in them in the literature. For instance, Cummings and Shelah [CuSh 541] investigate the natural generalizations $\mathfrak{b}_\lambda$ of the unbounded number and the dominating number $\mathfrak{d}_\lambda$, giving simple constraints on the triple of cardinals $(\mathfrak{b}_\lambda, \mathfrak{d}_\lambda, 2^\lambda)$ and proving that any triple of cardinals obeying these constraints can be realized. In a somewhat parallel work [ShSj 643], Shelah and Spasojevič study $\mathfrak{b}_\lambda$ and the generalization $\mathfrak{t}_\lambda$ of the tower number. Zapletal [Za97] investigated the splitting number $\mathfrak{s}_\lambda$ – here the situation is really complicated as the inequality $\mathfrak{s}_\lambda > \lambda^+$ needs large cardinals. And so on.

To decide if the various definitions lead to different and interesting cardinals we need a forcing technology parallel to that develoiped for CS iterations *with new reals.* So in this context the methods and results of [Sh 587], [Sh 667] are not very useful.

4.2. **Properness over diamonds.** It seems that the first somewhat general iteration theorem for iterations with uncountable supports *when new subsets of* $\lambda$ *are added* is presented in [RoSh 655]. In that paper a condition called *properness over D–semi diamonds* is defined for a normal filter $D$ on an uncountable cardinal $\lambda$ such that $\lambda=\lambda^{<\lambda}$. This property implies that the forcing notion is $\lambda$–proper and we do have a preservation theorem for it

**Theorem 4.2** (Rosłanowski and Shelah [RoSh 655, Theorem 2.7])**.** *Assume that* $\bar{\mathbb{Q}}=\langle\mathbb{P}_\alpha,\underaccent{\tilde}{\mathbb{Q}}_\alpha:\alpha<\zeta^*\rangle$ *is a* $\lambda$*–support iteration such that for each* $\alpha<\zeta^*$

$$\Vdash_{\mathbb{P}_\alpha} \text{“ } \underaccent{\tilde}{\mathbb{Q}}_\alpha \text{ is proper for } D\text{–semi diamonds ”.}$$

Then $\mathbb{P}_{\zeta^*}=\lim(\bar{\mathbb{Q}})$ *is proper for D–semi diamonds.*

There are forcing notions “for the $\lambda$–reals” which are proper over semi diamonds.

**Definition 4.3.** Let $\bar{\mu}=\langle\mu_\alpha:\alpha<\lambda\rangle$, $\mu_\alpha\leq\lambda$ are regular uncountable cardinals.

(1) A set $T\subseteq\bigcup\limits_{i<\lambda}\prod\limits_{\alpha<i}\mu_\alpha$ is *a complete* $\bar{\mu}$*–tree* if
  (a) $(\forall\eta\in T)(\exists\nu\in T)(\eta\lhd\nu)$,
  (b) $\eta\lhd\nu\in T \quad\Rightarrow\quad \eta\in T$,
  (c) if $\langle\eta_i:i<\delta\rangle\subseteq T$ is a $\lhd$–increasing chain, $\delta<\lambda$, then there is $\eta\in T$ such that $\eta_i\lhd\eta$ for all $i<\delta$.

(2) **A condition in** $\mathbb{D}_{\bar{\mu}}$ is a complete $\bar{\mu}$–tree $T$ such that
  (a) $(\forall\eta\in \mathrm{split}(T))(\mathrm{succ}_T(\eta)=\lambda)$,
  (b) $(\forall\eta\in T)(\exists\nu\in T)(\eta\lhd\nu\in\mathrm{split}(T))$,
  (c) **if** $\delta<\lambda$ is limit and a sequence $\langle\eta_i:i<\delta\rangle\subseteq\mathrm{split}(T)$ is $\lhd$–increasing, **then** $\eta=\bigcup\limits_{i<\delta}\eta_i\in\mathrm{split}(T)$.

  **The order of** $\mathbb{D}_{\bar{\mu}}$ is the reverse inclusion.

(3) The forcing notion $\mathbb{M}_{\bar{\mu}}$ is defined similarly, but (a) is replaced by
  (a') $(\forall\eta\in\mathrm{split}(T))(\mathrm{succ}_T(\eta)$ contains a club of $\mu_{\mathrm{lh}(\eta)})$.

**Proposition 4.4** (Rosłanowski and Shelah [RoSh 655, Proposition 3.8])**.**
$\mathbb{D}_{\bar{\mu}}$ *is proper over semi diamonds.*

But it is not clear if $\mathbb{M}_{\bar{\mu}}$ is proper over semi diamonds (though some of its variants are). Thinking about forcing notions of that type, properness over semi diamonds was modified to cover more examples and *fuzzy properness over diamonds* was introduced in [RoSh 777]. This conditions has a flavor[3] of a property weaker than the properness over $D$–semi diamonds of [RoSh 655]. But there is a serious price to pay — the iteration theorem is proven under the assumption that $\lambda$ is inaccessible, see below.

*Context* 4.5. (1) $\lambda$ is a strongly inaccessible cardinal,

(2) $\lambda^*>\lambda$ is a regular cardinal, $A\subseteq\mathcal{H}_{<\lambda}(\lambda^*)$, $W\subseteq[A]^{\lambda}$, and if $a\in W$, $w\in[a]^{<\lambda}$, $f:w\longrightarrow a$, then $f\in a$ (hence also $0\in a$ for $a\in W$),

[3] differences in technical details make it unclear if the “properness condition” of [RoSh 655] implies that of [RoSh 777]

(3) for every $x \in \mathcal{H}(\chi)$ there is a model $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ such that $|N| = \lambda$, ${}^{<\lambda}N \subseteq N$, $x \in N$ and $N \cap A \in W$,
(4) $D$ is a normal filter on $\lambda$ such that there is a $D$–diamond.

**Definition 4.6.** We say that $\bar{F} = \langle F_\delta : \delta \in S\rangle$ is *a convenient $D$–diamond* if

- $S \in D^+$ contains all successor ordinals below $\lambda$, $\lambda \setminus S$ is unbounded in $\lambda$, $0 \notin S$, and
- $F_\delta \in {}^\delta\delta$ for all $\delta \in S$,
- $(\forall f \in {}^\lambda\lambda)(\{\delta \in S : F_\delta \subseteq f\} \in D^+)$.

**Definition 4.7.** Let $\mathbb{P}$ be a forcing notion. *A $\lambda$–base for $\mathbb{P}$ over $W$* is a pair $(\mathfrak{R}, \bar{\mathfrak{Y}})$ such that

(a) $\mathfrak{R} \subseteq \mathbb{P} \times \lambda \times A$ is a relation such that
$$\text{if } (p, \delta, x) \in \mathfrak{R} \text{ and } p \leq_{\mathbb{P}} p', \text{ then } (p', \delta, x) \in \mathfrak{R},$$
(b) $\bar{\mathfrak{Y}} = \langle \mathfrak{Y}_a : a \in W\rangle$ where, for each $a \in W$, $\mathfrak{Y}_a : \lambda \longrightarrow [a]^{<\lambda}$,
(c) if $q \in \mathbb{P}$, $a \in W$, and $\delta < \lambda$ is a limit ordinal, then there are $p \geq_{\mathbb{P}} q$ and $x \in \mathfrak{Y}_a(\delta)$ such that $(p, \delta, x) \in \mathfrak{R}$.

If $\mathfrak{R}$ is understood and $(p, \delta, x) \in \mathfrak{R}$, then we may say *$p$ obeys $x$ at $\delta$*.

**Definition 4.8.** Let $\mathbb{P}$ be a forcing notion and let $(\mathfrak{R}, \bar{\mathfrak{Y}})$ be a $\lambda$–base for $\mathbb{P}$ over $W$. Also let a model $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ be such that $|N| = \lambda$, ${}^{<\lambda}N \subseteq N$, $a \stackrel{\text{def}}{=} N \cap A \in W$ and $\{\lambda, \mathbb{P}, D, \mathfrak{R}\} \in N$. Furthermore, let $h : \lambda \longrightarrow N$ be such that the range $\mathrm{Rng}(h)$ of the function $h$ includes $\mathbb{P} \cap N$ and let $\bar{F} = \langle F_\delta : \delta \in S\rangle$ be a $D$–pre-diamond sequence.

(1) Let $\bar{\mathcal{I}} = \langle \mathcal{I}_\alpha : \alpha < \lambda\rangle \subseteq N$ list all open dense subsets of $\mathbb{P}$ from $N$. A sequence $\bar{p} = \langle p_\alpha : \alpha < \delta\rangle$ of conditions from $\mathbb{P} \cap N$ of length $\delta \leq \lambda$ is called *$\bar{\mathcal{I}}$–exact* if
$$(\forall \xi < \delta)(\exists \alpha < \delta)(p_\alpha \in \mathcal{I}_\xi).$$
(2) We say that $\bar{F}$ is *a quasi $D$–diamond sequence for* $(N, h, \mathbb{P})$ if for some (equivalently: all) list $\bar{\mathcal{I}} = \langle \mathcal{I}_\alpha : \alpha < \lambda\rangle$ of all open dense subsets of $\mathbb{P}$ from $N$, for every $\leq_{\mathbb{P}}$–increasing sequence $\bar{p} = \langle p_\alpha : \alpha < \lambda\rangle \subseteq \mathbb{P} \cap N$ such that
$$E \stackrel{\text{def}}{=} \{\delta < \lambda : \langle p_\alpha : \alpha < \delta\rangle \text{ is } \bar{\mathcal{I}}\text{–exact }\} \in D$$
(equivalently: $\bar{p}$ is $\bar{\mathcal{I}}$–exact) we have
$$\{\delta \in E : (\forall \alpha < \delta)(h \circ F_\delta(\alpha) = p_\alpha)\} \in D^+.$$
(3) For a limit ordinal $\delta \in S$ we define $\mathcal{Y}(\delta) = \mathcal{Y}(N, \mathbb{P}, h, \bar{F}, \mathfrak{R}, \bar{\mathfrak{Y}}, \delta)$ as the set
$$\big\{x \in \mathfrak{Y}_a(\delta) : \begin{array}{l}\text{if } \langle h \circ F_\delta(\alpha) : \alpha < \delta\rangle \text{ is a } \leq_{\mathbb{P}}\text{–increasing sequence}\\ \text{of conditions from } \mathbb{P},\\ \text{then there is a condition } p \in \mathbb{P} \text{ such that}\\ (\forall \alpha < \delta)(h \circ F_\delta(\alpha) \leq_{\mathbb{P}} p) \text{ and } (p, \delta, x) \in \mathfrak{R}\big\}\end{array}$$
(4) Let $\bar{\mathcal{I}} = \langle \mathcal{I}_\alpha : \alpha < \lambda\rangle \subseteq N$ list all open dense subsets of $\mathbb{P}$ from $N$. A sequence $\bar{q} = \langle q_{\delta,x} : \delta \in S \text{ limit } \ \& \ x \in \mathcal{Y}(\delta)\rangle \subseteq N \cap \mathbb{P}$ is called *a fuzzy candidate over $\bar{F}$ for* $(N, h, \mathbb{P}, \mathfrak{R}, \bar{\mathfrak{Y}}, \bar{\mathcal{I}})$ whenever
 ($\alpha$) $\{\delta \in S : (\forall x \in \mathcal{X}_\delta)(q_{\delta,x} \in \mathcal{I}_\alpha)\} = S \mod D$ for each $\alpha < \lambda$, and
 ($\beta$) if $\delta \in S$ is a limit ordinal, $x \in \mathcal{X}_\delta$, and $\langle h \circ F_\delta(\alpha) : \alpha < \delta\rangle$ is a $\leq_{\mathbb{P}}$–increasing $\bar{\mathcal{I}}$–exact sequence of members of $\mathbb{P} \cap N$, then $(\forall \alpha < \delta)(h \circ F_\delta(\alpha) \leq_{\mathbb{P}} q_{\delta,x})$ and $(q_{\delta,x}, \delta, x) \in \mathfrak{R}$.

(5) Let $\bar{q} = \langle q_{\delta,x} : \delta \in S$ limit $\&\ x \in \mathcal{X}_\delta\rangle$ be a fuzzy candidate over $\bar{F}$ for $(N, h, \mathbb{P}, \mathfrak{R}, \bar{\mathfrak{Y}}, \bar{\mathcal{I}})$, and $r \in \mathbb{P}$. We define a game $\Game^{\rm fuzzy}_\lambda(r, N, \bar{\mathcal{I}}, h, \mathbb{P}, \bar{F}, \bar{q})$ of two players, *Generic* and *Antigeneric*, as follows. A play lasts $\lambda$ moves, in the $i^{\rm th}$ move a condition $r_i \in \mathbb{P}$ and a set $C_i \in D$ are chosen such that $(\forall j < i)(r \le r_j \le r_i)$, and Generic chooses $r_i, C_i$ if $i \in S = {\rm Dom}(\bar{F})$, and Antigeneric chooses $r_i, C_i$ if $i \notin S$. In the end Generic wins the play if

- ($\alpha$) $(\forall \alpha < \lambda)(\exists i < \lambda)(\exists p \in \mathbb{P} \cap N)(p \in \mathcal{I}_\alpha\ \&\ p \le r_i)$, and
- ($\beta$) if $\delta \in S \cap \bigcap\limits_{i<\delta} C_i$ is a limit ordinal, $\langle h \circ F_\delta(\alpha) : \alpha < \delta\rangle$ is a $\le_{\mathbb{P}}$–increasing $\bar{\mathcal{I}}$–exact sequence and $(\forall \alpha < \delta)(\exists i < \delta)(h \circ F_\delta(\alpha) \le r_i)$, then for some $x \in \mathcal{X}_\delta$ we have $q_{\delta,x} \le r_\delta$.

(6) Let $\bar{q}$ be a fuzzy candidate over $\bar{F}$ for $(N, h, \mathbb{P}, \mathfrak{R}, \bar{\mathfrak{Y}}, \bar{\mathcal{I}})$. We say that a condition $r \in \mathbb{P}$ is $(\mathfrak{R}, \bar{\mathfrak{Y}})$*–fuzzy generic for* $\bar{q}$ *(over* $(N, \bar{\mathcal{I}}, h, \mathbb{P}, \bar{F})$*)* if Generic has a winning strategy in the game $\Game^{\rm fuzzy}_\lambda(r, N, \bar{\mathcal{I}}, h, \mathbb{P}, \bar{F}, \bar{q})$.

(7) A $(<\lambda)$–complete forcing notion $\mathbb{P}$ is *fuzzy proper over quasi* $D$*–diamonds for* $W$ whenever for some $\lambda$–base $(\mathfrak{R}, \bar{\mathfrak{Y}})$ for $\mathbb{P}$ over $W$ and for some $c \in \mathcal{H}(\chi)$,

($\circledast$) *if*
- $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$, $|N| = \lambda$, ${}^{<\lambda}N \subseteq N$, $\lambda, \mathbb{P}, c, \mathfrak{R} \in N$, and $a \stackrel{\rm def}{=} N \cap A \in W$, $p \in \mathbb{P} \cap N$,
- $h : \lambda \longrightarrow N$ satisfies $\mathbb{P} \cap N \subseteq {\rm Rng}(h)$, and
- $\bar{F}$ is a quasi $D$–diamond for $(N, h, \mathbb{P})$ and $\bar{q}$ is a fuzzy candidate over $\bar{F}$,

*then* there is $r \in \mathbb{P}$ stronger than $p$ and such that $r$ is $(\mathfrak{R}, \bar{\mathfrak{Y}})$–fuzzy generic for $\bar{q}$.

**Theorem 4.9** (Rosłanowski and Shelah [RoSh 777, Theorem A.3.10])**.** *Let* $A, W, D$ *be as in 4.5 and let* $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underaccent{\tilde}{\mathbb{Q}}_\alpha : \alpha < \zeta^*\rangle$ *be a* $\lambda$*–support iteration of* $\lambda$*–complete forcing notions. Suppose also that for each* $\zeta < \zeta^*$ *we have* $\bar{\mathfrak{Y}}^\zeta$ *and* $\mathbb{P}_\zeta$*–names* $\underaccent{\tilde}{\mathfrak{R}}_\zeta, \underaccent{\tilde}{c}_\zeta$ *such that*

$$\Vdash_{\mathbb{P}_\zeta} \text{“ } \underaccent{\tilde}{\mathbb{Q}}_\zeta \text{ is fuzzy proper over quasi } D\text{–diamonds for } W \text{ with witnesses } (\underaccent{\tilde}{\mathfrak{R}}_\zeta, \bar{\mathfrak{Y}}^\zeta) \text{ and } \underaccent{\tilde}{c}_\zeta \text{”.}$$

*Then* $\mathbb{P}_{\zeta^*} = \lim(\bar{\mathbb{Q}})$ $\lambda$*–proper and even more.*

**Proposition 4.10.** *If* $\mu_\alpha < \lambda$ *(for each* $\alpha < \lambda$*) then* $\mathbb{M}_{\bar{\mu}}$ *is fuzzy proper over quasi diamonds.*

4.3. **Bounding properties.** The various variants of the properness over diamonds have flavours of bounding properties.

**Definition 4.11.** Let $\mathbb{Q}$ be a strategically $(<\lambda)$–complete forcing notion.

(1) For a condition $p \in \mathbb{Q}$ we define a game $\Game^{\rm rcA}_{\bar{\mu}}(p, \mathbb{Q})$ between two players, Generic and Antigeneric, as follows. A play of $\Game^{\rm rcA}_{\bar{\mu}}(p, \mathbb{Q})$ lasts $\lambda$ steps and during a play a sequence

$$\Big\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha\rangle : \alpha < \lambda\Big\rangle$$

is constructed. Suppose that the players have arrived to a stage $\alpha < \lambda$ of the game. Now,

$(\aleph)_\alpha$ first Generic chooses a non-empty set $I_\alpha$ of cardinality $< \mu_\alpha$ and a system $\langle p^\alpha_t : t \in I_\alpha\rangle$ of conditions from $\mathbb{Q}$,

$(\sqsupset)_\alpha$ then Antigeneric answers by picking a system $\langle q^\alpha_t : t\in I_\alpha\rangle$ of conditions from $\mathbb{Q}$ such that $(\forall t\in I_\alpha)(p^\alpha_t\leq q^\alpha_t)$.

At the end, Generic wins the play

$$\Big\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t\in I_\alpha\rangle : \alpha<\lambda\Big\rangle$$

of $\partial^{\rm rcA}_{\bar{\mu}}(p,\mathbb{Q})$ if and only if

$(\circledast)^{\rm rc}_{\rm A}$ there is a condition $p^*\in\mathbb{Q}$ stronger than $p$ and such that[4]

$$p^*\Vdash_{\mathbb{Q}} \text{“ } \big\{\alpha<\lambda : \big(\exists t\in I_\alpha\big)\big(q^\alpha_t\in\Gamma_{\mathbb{Q}}\big)\big\}=\lambda \text{ ”}.$$

(2) A game $\partial^{\rm rcB}_{\mathcal{U},\bar{\mu}}(p,\mathbb{Q})$ is defined similarly, except that the winning criterion $(\circledast)^{\rm rc}_{\rm A}$ is replaced by

$(\circledast)^{\rm rc}_{\rm B}$ there is a condition $p^*\in\mathbb{Q}$ stronger than $p$ and such that

$$p^*\Vdash_{\mathbb{Q}} \text{“ } \big\{\alpha<\lambda : \big(\exists t\in I_\alpha\big)\big(q^\alpha_t\in\Gamma_{\mathbb{Q}}\big)\big\}\in\mathcal{U}^{\mathbb{Q}} \text{ ”}.$$

(3) For a condition $p\in\mathbb{Q}$ we define a game $\partial^{\rm rc\mathbf{b}}_{\mathcal{U},\bar{\mu}}(p,\mathbb{Q})$ between Generic and Antigeneric as follows. A play of $\partial^{\rm rc\mathbf{b}}_{\mathcal{U},\bar{\mu}}(p,\mathbb{Q})$ lasts $\lambda$ steps and during a play a sequence

$$\Big\langle \zeta_\alpha, \langle p^\alpha_\xi, q^\alpha_\xi : \xi<\zeta_\alpha\rangle : \alpha<\lambda\Big\rangle$$

is constructed. Suppose that the players have arrived to a stage $\alpha<\lambda$ of the game. Now, Generic chooses a non-zero ordinal $\zeta_\alpha<\mu_\alpha$ and then the two players play a subgame of length $\zeta_\alpha$ alternatively choosing successive terms of a sequence $\langle p^\alpha_\xi, q^\alpha_\xi : \xi<\zeta_\alpha\rangle$. At a stage $\xi<\zeta_\alpha$ of the subgame, first Generic picks a condition $p^\alpha_\xi\in\mathbb{Q}$ and then Antigeneric answers with a condition $q^\alpha_\xi$ stronger than $p^\alpha_\xi$.

At the end, Generic wins the play

$$\Big\langle \zeta_\alpha, \langle p^\alpha_\xi, q^\alpha_\xi : \xi<\zeta_\alpha\rangle : \alpha<\lambda\Big\rangle$$

of $\partial^{\rm rc\mathbf{b}}_{\mathcal{U},\bar{\mu}}(p,\mathbb{Q})$ if and only if

$(\circledast)^{\rm rc}_{\mathbf{b}}$ there is a condition $p^*\in\mathbb{Q}$ stronger than $p$ and such that

$$p^*\Vdash_{\mathbb{Q}} \text{“ } \big\{\alpha<\lambda : \big(\exists \xi<\zeta_\alpha\big)\big(q^\alpha_\xi\in\Gamma_{\mathbb{Q}}\big)\big\}\in\mathcal{U}^{\mathbb{Q}} \text{ ”}.$$

(4) A game $\partial^{\rm rc\mathbf{a}}_{\bar{\mu}}(p,\mathbb{Q})$ is defined similarly except that the winning criterion $(\circledast)^{\rm rc}_{\mathbf{b}}$ is changed so that "$\in\mathcal{U}^{\mathbb{Q}}$" is replaced by "$=\lambda$".

(5) We say that a forcing notion $\mathbb{Q}$ is *reasonably A–bounding over* $\bar{\mu}$ if
 (a) $\mathbb{Q}$ is strategically $({<}\lambda)$–complete, and
 (b) for any $p\in\mathbb{Q}$, Generic has a winning strategy in the game $\partial^{\rm rcA}_{\bar{\mu}}(p,\mathbb{Q})$.

In an analogous manner we define when the forcing notion $\mathbb{Q}$ is *reasonably X–bounding over* $\mathcal{U},\bar{\mu}$ (for $\mathrm{X}\in\{\mathrm{B},\mathbf{a},\mathbf{b}\}$) — just using the game $\partial^{\rm rcX}_{\mathcal{U},\bar{\mu}}(p,\mathbb{Q})$ appropriately.

**Theorem 4.12** (Rosłanowski and Shelah [RoSh 860, Theorems 2.5, 2.10])**.** *Assume that*

(a) *$\lambda$ is a strongly inaccessible cardinal,*
(b) *$\bar{\mu}=\langle\mu_\alpha:\alpha<\lambda\rangle$, each $\mu_\alpha$ is a regular cardinal satisfying (for $\alpha<\lambda$)*

$$\aleph_0\leq\mu_\alpha\leq\lambda \qquad \textit{and} \qquad \big(\forall f\in{}^\alpha\mu_\alpha\big)\big(\big|\prod_{\xi<\alpha}f(\xi)\big|<\mu_\alpha\big),$$

[4] equivalently, for every $\alpha<\lambda$ the set $\big\{q^\alpha_t : t\in I_\alpha\big\}$ is pre-dense above $p^*$

(c) *$\mathcal{U}$ is a normal filter on $\lambda$ and $X \in \{A, B\}$.*

*Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\xi : \xi < \gamma \rangle$ be a $\lambda$–support iteration such that for every $\xi < \gamma$,*

$$\Vdash_{\mathbb{P}_\xi} \text{“ } \underset{\sim}{\mathbb{Q}}_\xi \text{ is reasonably } X\text{–bounding over } \mathcal{U}, \bar{\mu} \text{ ”.}$$

*Then $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ is reasonably $\mathbf{x}$–bounding over $\mathcal{U}, \bar{\mu}$ (and so also $\lambda$–proper).*

There are interesting consequences of the above theorem for cardinal characteristics of the $\lambda$–reals. It was shown in Cummings and Shelah [CuSh 541] that $\mathfrak{d}_\lambda = \mathfrak{d}_\lambda$club (whenever $\lambda > \beth_\omega$ is regular). The following conclusion is an interesting addition to that result.

*Conclusion* 4.13 (Rosłanowski and Shelah [RoSh 860, Conclusion 4.11]). It is consistent that $\lambda$ is an inaccessible cardinal and there are two normal filters $D', D''$ on $\lambda$ such that $\mathfrak{d}_\lambda^{D'} \neq \mathfrak{d}_\lambda^{D''}$ (see 4.1).

The next result is of interest as it shows that the $\lambda$–versions of cardinal characteristics of the reals may behave totally differently from their “ancestors”:

*Conclusion* 4.14 (Rosłanowski and Shelah [RoSh 777, Proposition B.8.5]). It is consistent that $c(\bar{\eta}) < c(\bar{\mu})$ (see 4.1) for some $\bar{\mu} = \langle \mu_\alpha : \alpha < \lambda \rangle$, $\bar{\eta} = \langle \eta_\alpha : \alpha < \lambda \rangle$ such that $\mu_\alpha < \eta_\alpha < \lambda$ (for $\alpha < \lambda$).

4.4. **What's cooking?** In search for good questions that could stimulate the development of the theory of forcing iterated with uncountable supports, Shelah introduced *reasonable ultrafilters* in [Sh 830].

**(Non-)paid commercial:**

*Shelah's* [Sh 830] *is a very nicely written paper, full of details, short and practically self-contained. Proofs there are presented in full detail and it is a* must read *paper!*

In forthcoming [RoSh 890] we will present a proof that A–complete forcing notions preserve a special kind of reasonable ultrafilters and we will use it to show that, consistently, a reasonable ultrafilter on an inaccessible $\lambda$ may have a small generating system.

Less ready but very possible upcoming results include:

- first steps towards iteration theorems covering forcing notions like $\mathbb{M}_{\bar{\mu}}$ where $\mu_\alpha = \lambda$ (for each $\alpha < \lambda$, $\lambda$ inaccessible),
- first steps towards theorems on not-adding Cohen reals.

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